Joint Design of Information and Transfers in a Team Production Problem

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Game Theory I

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Papers

- Halac, M., E. Lipnowski, and D. Rappoport (2021). "Rank Uncertainty in Organizations," American Economic Review 111, 757-86.
- Morris, S., D. Oyama, and S. Takahashi (2022). "On the Joint Design of Information and Transfers."

Model

- Team project by agents $I = \{1, \ldots, |I|\}$ ($\mathcal{I} = 2^I$, $\mathcal{I}_{-i} = 2^{I \setminus \{i\}}$)
- Effort level $a_i \in \{0, 1\}$
- c_i: Agent i's cost of effort
- $P\colon \mathcal{I} \to [0, 1]$

  $P(S)$: Success probability when i works if and only if $i \in S$
- Assumptions:
  - Monotonicity: $P(S) < P(S')$ whenever $S \subsetneq S'$
  - Increasing returns to scale (IRS):
    $P(S) + P(S') \leq P(S \cup S') + P(S \cap S')$ for all $S, S \in \mathcal{I}$

    (P is a *convex game* if viewed as a cooperative game.)
- For each $i \in I$ and $S \in \mathcal{I}_{-i}$, denote

$$\Delta_i P(S) = P(S \cup \{i\}) - P(S).$$

Incentive Contracts

- Principal offers private contracts to agents to implement action profile I (all players exerting effort)

  as a smallest—hence unique—equilibrium outcome.

- Moral hazard (hidden action):
  Only the final outcome of the project is contractible.

  Bonus $b_i \geq 0$ paid to each agent upon success

- Ex post payoffs:

  $$\begin{cases} P(S \cup \{i\})b_i - c_i & \text{if } a_i = 1 \\ P(S)b_i & \text{if } a_i = 0 \end{cases}$$

- Payoff gain function:

  $$d_i(S) = \Delta_i P(S) - c_i$$

Incentive Schemes

Incentive scheme $\varphi = (\mathcal{T}, B)$:

- Type space $\mathcal{T} = ((T_i)_{i \in I}, \pi)$:
  - T_i: countable set of i's types $\quad (T = \prod_i T_i,\ T_{-i} = \prod_{j \neq i} T_j)$
  - $\pi \in \Delta(T)$: common prior
  - Assume $\pi_i(t_i) = \sum_{t_{-i}} \pi(t_i, t_{-i}) > 0$ for all i and t_i.
  - Write $\pi_i(t_{-i}|t_i) = \frac{\pi(t_i, t_{-i})}{\pi_i(t_i)}$.
- Bonus rule $B_i \colon T_i \to \mathbb{R}_+$: Bonus paid to agent i of type t_i
- Payoffs to agent i of type t_i:
  - $\sum_{t_{-i}} \pi_i(t_{-i}|t_i) P(S(\sigma_{-i}(t_{-i})) \cup \{i\}) B_i(t_i) - c_i$ if $a_i = 1$
  - $\sum_{t_{-i}} \pi_i(t_{-i}|t_i) P(S(\sigma_{-i}(t_{-i}))) B_i(t_i)$ if $a_i = 0$

Principal's Objective

- Incentive scheme $\varphi = (\mathcal{T}, B)$ *uniquely implements work* (or φ is a *UIW scheme*) if "always work" is the unique equilibrium of the Bayesian game induced by $(\mathcal{T}, B + \varepsilon)$ for every $\varepsilon > 0$.

- Total bonus minimization problem:

$$TB^* = \inf_{\varphi:\text{UIW}} TB(\varphi),$$

where

$$\begin{aligned} TB(\varphi) &= \sum_t \pi(t) P(I) \sum_i B_i(t_i) \\ &= P(I) \sum_i \sum_{t_i} \left(\sum_{t_{-i}} \pi(t) \right) B_i(t_i) \\ &= P(I) \sum_i \sum_{t_i} \pi_i(t_i) B_i(t_i). \end{aligned}$$

Results

1. Obtain a lower bound of $\sum_{t_i} \pi_i(t_i) B_i(t_i)$ for each UIW scheme $(\mathcal{T}, B)$.

   We provide a proof similar to that of Theorem 1(1) of MOT.

2. TB^* is bounded below by $\sum_i b_i^*$,

   where $b^* = (b_i^*)_{i \in I}$ is the unique solution to

   - $\min \sum_i b_i$
   - subject to the constraint that I satisfies sequential obedience in the complete information game given by the bonus profile b.

3. $\sum_i b_i^*$ is attained in the limit of some sequence of ε-elaborations of the complete information game given by the bonus profile b^*.

   Follows from the construction in the proof of Theorem 2 of OT.

4. The limit bonus distribution of any optimal sequence of UIW schemes is the degenerate distribution on b^*.
5. Structure of optimal limit bonus profile

   We derive the results using some known results from cooperative game theory (Shapley 1971; Hokari 2002).

Lower Bound of Expected Bonus Payment

- Π: Set of permutations of I
- $S_{-i}(\gamma)$: Set of agents that appear before i in $\gamma \in \Pi$
- For $i \in I$ and $\rho \in \Delta(\Pi)$, define

$$h_i(\rho) = \frac{c_i}{\sum_{\gamma \in \Pi} \rho(\gamma) \Delta_i P(S_{-i}(\gamma))}.$$

$\cdots$ Convex function of ρ

Proposition 1

For any UIW scheme $(\mathcal{T}, B)$ there exists $\rho \in \Delta(\Pi)$ such that

$$\sum_{t_i} \pi_i(t_i) B_i(t_i) > h_i(\rho)$$

for all $i \in I$.

Proof

Similar to the proof of Theorem 1(1) of MOT:

- Let $(\mathcal{T}, B)$ be a UIW scheme.
- Starting with the smallest strategy $\sigma_i^0(t_i) = 0$ for all $i = I$ and all $t_i \in T_i$, apply sequential best response in the order $1, 2, \ldots, |I|$.
- Let $\{\sigma^n\}$ be the obtained sequence of strategy profiles:
  - $\sigma_i^n(t_i) = 1$ if $i \equiv n \pmod{|I|}$ and $\sum_{t_{-i}} \pi_i(t_{-i}|t_i) B_i(t_i) \Delta_i P(S(\sigma_{-i}^{n-1}(t_{-i}))) > c_i$,
  - $\sigma_i^n(t_i) = \sigma_i^{n-1}(t_i)$ otherwise.
- By supermodularity, for each $i \in I$ and $t_i \in T_i$, $\{\sigma_i^n(t_i)\}$ is monotone increasing and converges to 1.

- Let $n_i(t_i) = n$ if $\sigma_i^{n-1}(t_i) = 0$ and $\sigma_i^n(t_i) = 1$.
  Write $n(t) = (n_1(t_1), \ldots, n_{|I|}(t_{|I|}))$.
- For $\gamma = (i_1, \ldots, i_{|I|}) \in \Pi$, let

$$T(\gamma) = \{t \in T \mid n_{i_1}(t_{i_1}) < \cdots < n_{i_{|I|}}(t_{|I|})\}.$$

- Define $\rho \in \Delta(\Pi)$ and $\rho_i(\cdot|t_i) \in \Delta(\Pi)$ for each $i \in I$ and $t_i \in T_i$ by

$$\rho(\gamma) = \sum_{t \in T(\gamma)} \pi(t),$$

$$\rho_i(\gamma|t_i) = \sum_{t_{-i}:(t_i,t_{-i}) \in T(\gamma)} \pi_i(t_{-i}|t_i).$$

- Note that $\rho(\gamma) = \sum_{t_i \in T_i} \pi_i(t_i)\rho_i(\gamma|t_i)$ for any $i \in I$.

- For any $i \in I$ and $t_i \in T_i$,

$$\begin{aligned}
c_i &< \sum_{t_{-i}} \pi_i(t_{-i}|t_i) B_i(t_i) \Delta_i P(S(\sigma_{-i}^{n_i(t_i)-1}(t_{-i}))) \\
&= \sum_{\gamma} \sum_{t_{-i}:(t_i,t_{-i}) \in T(\gamma)} \pi_i(t_{-i}|t_i) B_i(t_i) \Delta_i P(S_{-i}(\gamma)) \\
&= \sum_{\gamma} \rho_i(\gamma|t_i) B_i(t_i) \Delta_i P(S_{-i}(\gamma)).
\end{aligned}$$

- Therefore, for any $i \in I$ and $t_i \in T_i$,

$$B_i(t_i) > h_i(\rho_i(\cdot|t_i)),$$

where

$$h_i(\rho') = \frac{c_i}{\sum_{\gamma} \rho'(\gamma) \Delta_i P(S_{-i}(\gamma))},$$

which is a convex function of $\rho' \in \Delta(\Pi)$.

- Therefore,

$$\sum_{t_i} \pi_i(t_i) B_i(t_i) > \sum_{t_i} \pi_i(t_i) h_i(\rho_i(\cdot|t_i)).$$

- But by the convexity of h_i, we have

$$\sum_{t_i} \pi_i(t_i) h_i(\rho_i(\cdot|t_i)) \geq h_i \left(\sum_{t_i} \pi_i(t_i) \rho_i(\cdot|t_i) \right) = h_i(\rho)$$

  by Jensen's inequality.

- Therefore, we have

$$\sum_{t_i} \pi_i(t_i) B_i(t_i) > h_i(\rho).$$

Lower Bound of TB^*

- Since

$$\begin{aligned} TB((\mathcal{T}, B)) &= \sum_t \pi(t) P(I) \sum_i B_i(t_i) \\ &= P(I) \sum_i \sum_{t_i} \left(\sum_{t_{-i}} \pi(t) \right) B_i(t_i) \\ &= P(I) \sum_i \sum_{t_i} \pi_i(t_i) B_i(t_i), \end{aligned}$$

we have

$$TB((\mathcal{T}, B)) > P(I) \sum_{i \in I} h_i(\rho)$$

by Proposition 1.

- Consider the optimization problem

$$\min_{b \in \mathbb{R}^I_{++}} \sum_{i \in I} b_i$$

subject to the condition that there exists $\rho \in \Delta(\Pi)$ such that $b_i \geq h_i(\rho)$ for all $i \in I$, or

$$\sum_{\gamma} \rho(\gamma) \mathit{\Delta}_i P(S_{-i}(\gamma)) - \frac{c_i}{b_i} \geq 0 \text{ for all } i \in I. \qquad (*)$$

- By the strict convexity, this problem has a unique solution b^*.
- Since $b = (h_i(\rho))_{i \in I}$ trivially satisfies the constraint $b_i \geq h_i(\rho)$, we have $\sum_{i \in I} h_i(\rho) \geq \sum_{i \in I} b_i^*$.
- Therefore,

$$\inf TB((\mathcal{T}, B)) \geq P(I) \sum_{i \in I} b_i^*.$$

Sequential Obedience, Coalitional Obedience

- Condition $(*)$ is equivalent to sequential obedience of action profile $\mathbf{1}$ in the complete information BAS game defined by

$$d_i(a_{-i}; b_i) = \mathit{\Delta}_i P(S(a_{-i})) - \frac{c_i}{b_i}.$$

- This game is a potential game with a potential

$$\Phi(a; b) = P(S(a)) - \sum_{i \in S(a)} \frac{c_i}{b_i}.$$

- Therefore, by MOT, condition $(*)$ is equivalent to coalitional obedience of $\mathbf{1}$: $\Phi(\mathbf{1}; b) \geq \Phi(a; b)$ for all $a \in A$, or

$$\sum_{i \in I \setminus S} \frac{c_i}{b_i} \leq P(I) - P(S) \text{ for all } S \in \mathcal{I}. \qquad (**)$$

Proposition 2

$\inf_{(\mathcal{T},B)} TB((\mathcal{T},B)) = P(I)\sum_{i\in I} b_i^*$.
In particular, for any $\varepsilon > 0$, there exists an ε'-elaboration $(\mathcal{T},B)$ of $(d_i(\cdot;b_i^* + \varepsilon/[2|I|P(I)]))_{i\in I}$ such that $TB((\mathcal{T},B)) \leq P(I)\sum_{i\in I} b_i^* + \varepsilon$.

- Follows from the construction in Theorem 2 of OT.

Proof

- Let b^*, ρ^* be the solution.
- For each $i \in I$, let $\bar{b}_i > \dfrac{c_i}{\Delta_i P(\emptyset)}$.
- Fix any $\varepsilon > 0$.
- Let $\eta > 0$ be such that

$$\sum_{S \in \mathcal{I}_{-i}} (1-\eta)^{|S|} \rho^*(\{\gamma \in \Pi \mid S_{-i}(\gamma) = S\}) \Delta_i P(S) - \frac{c_i}{b_i^* + \frac{\varepsilon}{2|I|P(I)}} > 0 \quad (1)$$

and

$$1 - (1-\eta)^{|I|-1} \leq \frac{\varepsilon}{2P(I)\left(\sum_{i \in I} \bar{b}_i - \sum_{i \in I} b_i^*\right)}. \quad (2)$$

- Write $\varepsilon' = 1 - (1-\eta)^{|I|-1}$.

- Construct the information structure $\mathcal{T}$ as follows:
  - $T_i = \{1, 2, \ldots\}$
  - m drawn from $\mathbb{Z}_+$ according to the distribution $\eta(1-\eta)^m$.
  - γ drawn from Π according to ρ^*.
  - Player i receives signal t_i given by

$$t_i = m + (\text{ranking of } i \text{ in } \gamma).$$

- Define the bonus rule B by

$$B_i(t_i) = \begin{cases} \bar{b}_i & \text{if } t_i \leq |I| - 1, \\ b_i^* + \frac{\varepsilon}{2|I|P(I)} & \text{if } t_i \geq |I|. \end{cases}$$

- $(\mathcal{T}, B)$ is an ε'-elaboration of $(d_i(\cdot; b_i^* + \varepsilon/[2|I|P(I)]))_{i \in I}$, where $\eta = 1 - (1-\varepsilon')^{1/(|I|-1)}$.

- In this elaboration, in any strategy surviving iterative dominance, all types t_i play action 1:
  - By construction, types $t_i \leq |I| - 1$ play the dominant action 1.
  - If types $t_j < \tau$ play action 1,
    then the payoff for type $t_i = \tau$ is at least

$$\sum_{S \in \mathcal{I}_{-i}} (1-\eta)^{|S|} \rho^*(\{\gamma \in \Pi \mid S_{-i}(\gamma) = S\}) \\ \times d_i \left(a(S); b_i^* + \tfrac{\varepsilon}{2|I|P(I)} \right) \times (\text{constant}) > 0.$$

- Therefore,

$$TB((\mathcal{T}, B)) \leq \varepsilon' P(I) \sum_{i \in I} \bar{b}_i + (1-\varepsilon') P(I) \sum_{i \in I} \left(b_i^* + \tfrac{\varepsilon}{2|I|P(I)} \right) \\ \leq P(I) \sum_{i \in I} b_i^* + \varepsilon.$$

Limit Bonus Distribution

- Let $\{(\mathcal{T}^k, B^k)\}$ be an optimal sequence of UIW schemes:

  i.e., a sequence of UIW schemes such that $TB((\mathcal{T}^k, B^k)) \to P(I) \sum_{i \in I} b_i^*$ as $k \to \infty$.

Proposition 3

For each $i \in I$, B_i^k converges to b_i^* weakly (or, in distribution) as $k \to \infty$.

- I.e., for all $\delta > 0$,

$$\pi_i^k(\{t_i \mid |B_i^k(t_i) - b_i^*| < \delta\}) \to 1$$

  as $k \to \infty$.

Lemma 1

For any $i \in I$ and any distribution τ on $\Delta(\Pi)$,

$$\int h_i(\rho) d\tau(\rho) \geq h_i\left(\int \rho d\tau(\rho)\right),$$

with equality only if the τ-distribution of h_i is degenerate.

- $x \mapsto \frac{1}{x}$ is strictly convex, and $\frac{1}{h_i(\rho)}$ is linear in ρ.
- Therefore, by Jensen's inequality,

$$\int h_i(\rho) d\tau(\rho) = \int \frac{1}{\frac{1}{h_i(\rho)}} d\tau(\rho) \geq \frac{1}{\int \frac{1}{h_i(\rho)} d\tau(\rho)}$$
$$= \frac{1}{\frac{1}{h_i\left(\int \rho d\tau(\rho)\right)}} = h_i\left(\int \rho d\tau(\rho)\right),$$

  with equality only if the τ-distribution of $\frac{1}{h_i}$, hence of h_i, is degenerate.

Proof of Proposition 3

- Take any sequence $\{(\mathcal{T}^k, B^k)\}$ such that $TB^k = TB((\mathcal{T}^k, B^k)) \to TB^* = P(I) \sum_{i \in I} b_i^*$.
- Let ρ^k and ρ_i^k, $i \in I$, be as in the proof of Proposition 1.

Claim 1

For all $i \in I$, $\sum_{t_i} \pi_i^k(t_i)|B_i^k(t_i) - h_i(\rho_i^k(\cdot|t_i))| \to 0$ as $k \to 0$.

- We have

$$\begin{aligned}
&\sum_i \sum_{t_i} \pi_i^k(t_i)|B_i^k(t_i) - h_i(\rho_i^k(\cdot|t_i))| \\
&= \sum_i \sum_{t_i} \pi_i^k(t_i)(B_i^k(t_i) - h_i(\rho_i^k(\cdot|t_i))) \\
&\leq \sum_i \sum_{t_i} \pi_i^k(t_i)B_i^k(t_i) - \sum_i h_i(\rho^k) \\
&\leq \sum_i \sum_{t_i} \pi_i^k(t_i)B_i^k(t_i) - \sum_i b_i^* \to 0
\end{aligned}$$

as $k \to 0$.

- Take any subsequence of $\{(\mathcal{T}^k, B^k)\}$ (again denoted by $\{(\mathcal{T}^k, B^k)\}$).

  We want to show that it has a subsequence (again denoted by $\{(\mathcal{T}^k, B^k)\}$) such that for each $i \in I$, B_i^k converges to b_i^* weakly.

- For each $i \in I$, write τ_i^k for the distribution of ρ_i^k on $\Delta(\Pi)$.

- Since the support of τ_i^k is contained in the compact set $\Delta(\Pi)$, we can take a subsequence such that for each $i \in I$, τ_i^k converges to some τ_i^* weakly as $k \to \infty$ (by Prokhorov's Theorem).

Claim 2

For all $i \in I$, $h_i(\rho_i^k) \to b_i^*$ weakly as $k \to \infty$.

- For all $i \in I$,

$$\int \rho d\tau_i^k = \sum_{t_i} \pi_i^k(t_i)\rho_i^k(\cdot|t_i) = \rho^k.$$

- By the boundedness of $\Delta(\Pi)$, letting $k \to \infty$ we have

$$\int \rho d\tau_i^* = \rho^*$$

  for all $i \in I$, where $\rho^* = \lim_{k\to\infty} \rho^k$.

- Then we have

$$\sum_i \left| \int h_i(\rho) d\tau_i^k - h_i\left(\int \rho d\tau_i^k\right) \right|$$
$$= \sum_i \left(\int h_i(\rho) d\tau_i^k - h_i\left(\int \rho d\tau_i^k\right)\right)$$
$$= \frac{TB^k}{P(I)} - \sum_i h_i(\rho^k) \leq \frac{TB^k}{P(I)} - \sum_i b_i^* \to 0 \text{ as } k \to \infty.$$

- Therefore, for all $i \in I$,

$$\int h_i(\rho) d\tau_i^* = h_i\left(\int \rho d\tau_i^*\right) = h_i(\rho^*),$$

  and $\sum_i h_i(\rho^*) = \sum_i b_i^*$, and hence $h_i(\rho^*) = b_i^*$ by the uniqueness of the solution.

- Therefore, by Lemma 1, the τ_i^*-distribution of h_i is degenerate on b_i^*.

- Now fix any $\delta > 0$.
- Since $|B_i^k - b_i^*| \geq \delta$ implies $|B_i^k - h_i(\rho_i^k)| \geq \delta/2$ or $|h_i(\rho_i^k) - b_i^*| \geq \delta/2$,

$$\pi_i^k(|B_i^k - b_i^*| \geq \delta)$$
$$\leq \pi_i^k(|B_i^k - h_i(\rho_i^k)| \geq \delta/2) + \pi_i^k(|h_i(\rho_i^k) - b_i^*| \geq \delta/2).$$

- Let $k \to \infty$. Then, by Claim 1,

$$\pi_i^k(|B_i^k - h_i(\rho_i^k)| \geq \delta/2)\delta/2$$
$$\leq \sum_{t_i} \pi_i^k(t_i)|B_i^k(t_i) - h_i(\rho_i^k(\cdot|t_i))| \to 0,$$

while by Claim 2,

$$\pi_i^k(|h_i(\rho_i^k) - b_i^*| \geq \delta/2) \to 0.$$

Comparative Statics

- Without loss, we assume that $P(\emptyset) = 0$.
- Let b^* be the optimal limit bonus profile, and $\rho^* \in \Delta(\Pi)$ an associated ordered outcome.

  Write $x_i^* = \frac{c_i}{b_i^*}$.
- $x^* = (x_1^*, \ldots, x_{|I|}^*)$ satisfies the sequential obedience condition with equality:

$$x_i^* = \sum_{\gamma} \rho^*(\gamma) \mathit{\Delta}_i P(S_{-i}(\gamma)) \text{ for all } i \in I. \qquad (\star)$$

- $x^* = (x_1^*, \ldots, x_{|I|}^*)$ satisfies the sequential obedience condition with equality:

$$x_i^* = \sum_{\gamma} \rho^*(\gamma) \Delta_i P(S_{-i}(\gamma)) \text{ for all } i \in I. \qquad (\star)$$

- For each $\gamma \in \Pi$, define $\alpha^\gamma = (\alpha_1^\gamma, \ldots, \alpha_{|I|}^\gamma)$ by

$$\alpha_i^\gamma = \Delta_i P(S_{-i}(\gamma)).$$

- $(\star)$ says:

  x^* is written as a convex combination of $(\alpha^\gamma)_{\gamma \in \Pi}$, where coefficients are given by ρ^*.

Some Facts from Cooperative Game Theory

From Shapley (1971), "Cores of Convex Games," IJGT.

- P viewed as a cooperative game, the core of P is the set

$$C = \{x \in \mathbb{R}^I \mid x(I) = P(I),\ x(S) \geq P(S) \text{ for all } S \in \mathcal{I}\},$$

  where $x(S) = \sum_{i \in S} x_i$.

- C is a polytope (intersection of finitely many half spaces, bounded).

- $C \neq \emptyset$ by the convexity of P.

  In fact, $\alpha^\gamma \in C$ for all $\gamma \in \Pi$.

- $(\alpha^\gamma)_{\gamma \in \Pi}$ are precisely the *vertices* of C.

  These are all distinct by the strict convexity of P.

- Thus, x satisfies $(\star)$ if and only if $x \in C$.

- For $S \in \mathcal{I}$, let

  $$C_S = \{x \in C \mid x(S) = P(S)\}.$$

  (Define $C_\emptyset = C$.)

  These are the *faces* of C.

- The vertices of C_S are precisely the points α^γ such that agents in S are ranked higher in γ.

- For $x \in C$, let $\mathcal{S}_x = \{S \in \mathcal{I} \mid x \in C_S\}$.

  The members of $\mathcal{S}_x$ are nested:
  If $S, S' \in \mathcal{S}_x$, then $S \subset S'$ or $S' \subset S$.

Structure of Optimal Bonus Profile and Associated Ordered Outcomes

- Let b^* be the optimal limit bonus profile, and ρ^* an associated ordered outcome.

  Let $x^* = (c_i/b_i^*)_{i \in I}$ as before: $x^* = \sum_{\gamma \in \Pi} \rho^*(\gamma)\alpha^\gamma$.

- x^* is the unique solution to the problem:

  $$\min \sum_{i \in I} \frac{c_i}{x_i}$$

  subject to

  $$x \in C, \qquad (\star\star)$$

  that is,

  $$x(I) = P(I),$$
  $$x(S) \geq P(S) \quad (S \in \mathcal{I}).$$

- Suppose we know that $\mathcal{S}_{x^*} = \{S_0^*, S_1^*, \ldots, S_L^*\}$ where $\emptyset = S_0^* \subset S_1^* \subset \cdots \subset S_L^* = I$.
- In the optimization problem, the constraints $x(S_\ell^*) = P(S_\ell^*)$, $\ell = 1, \ldots, L$, are precisely the binding constraints.
- For any γ such that $\rho^*(\gamma) > 0$,

$$\gamma = (S_1^*, S_2^* \setminus S_1^*, \ldots, S_L^* \setminus S_{L-1}^*).$$

- Define the weak order $\succsim$ on I by
  - $i \succ i'$ if and only if $i \in S_\ell^* \setminus S_{\ell-1}^*$ and $i' \in S_{\ell'}^* \setminus S_{\ell'-1}^*$ for some $\ell < \ell'$; and
  - $i \sim i'$ if and only if $i, i' \in S_\ell^* \setminus S_{\ell-1}^*$ for some ℓ.

Extreme Cases

- $i \sim i'$ for all $i, i' \in I$ (i.e., $L = 1$) if and only if

$$\frac{P(I)}{\sum_{i \in I} \sqrt{c_i}} > \frac{P(S)}{\sum_{i \in S} \sqrt{c_i}}$$

for all $S \neq I$.

- $i \succ i'$ or $i' \succ i$ whenever $i \neq i'$ (i.e., $L = |I|$, or $x^* = \alpha^\gamma$ for some $\gamma \in \Pi$) if and only if there exists $\gamma = (i_1, \ldots, i_{|I|}) \in \Pi$ such that

$$\frac{\Delta_{i_{k+1}} P(S_{-i_{k+1}}(\gamma))}{\sqrt{c_{i_{k+1}}}} \leq \frac{\Delta_{i_k} P(S_{-i_k}(\gamma))}{\sqrt{c_{i_k}}}$$

for all $k = 1, \ldots, |I| - 1$.

Connection to Cooperative Game Theory

- Recall the problem:

$$\min \sum_{i \in I} \frac{c_i}{x_i} \qquad \text{subject to} \quad x \in C$$

- This is a well-studied problem in cooperative game theory.
- When $c_1 = \cdots = c_{|I|}$, the solution x^* coincides with the "constrained egalitarian allocation" of Dutta and Ray (1989), or the "Dutta-Ray solution".
- For general c_i's, x^* is a special case of a generalized Dutta-Ray solution (e.g., Hokari (2002)).
- By Hokari (2002), the solution x^* is explicitly written as

$$x_i^* = \max_{S \subset I, S \ni i} \min_{T \subset S \setminus \{i\}} \frac{\sqrt{c_i}(P(S) - P(T))}{\sum_{j \in S \setminus T} \sqrt{c_j}}.$$

Proposition 4

The unique limit optimal bonus profile $b^* = (b_i^*)_{i \in I}$ *is given by*

$$b_i^* = \min_{S \subset I, S \ni i} \max_{T \subset S \setminus \{i\}} \frac{\sqrt{c_i} \sum_{j \in S \setminus T} \sqrt{c_j}}{P(S) - P(T)}.$$

Proposition 5

- b_i^* *is strictly increasing in* c_i.
- b_i^* *is increasing in* c_j, $j \neq i$.
- $\frac{b_i^*}{c_i}$ *is decreasing in* c_i.
- $\frac{b_i^*}{c_i}$ *is increasing in* c_j, $j \neq i$.