Robustness to Incomplete Information

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Game Theory I

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Papers

- Kajii, A. and S. Morris (1997). "The Robustness of Equilibria to Incomplete Information," Econometrica 65, 1283-1309.
- Kajii, A. and S. Morris (2020). "Refinements and Higher Order Beliefs: A Unified Survey," Japanese Economic Review 71, 7-34.

Robustness of Equilibria

- An analyst analyzes some strategic situation with a complete information game $\mathbf{g}$ and a Nash equilibrium a^* thereof.
- He knows that it is a good approximation, but he also thinks that there may be "small" payoff uncertainty among players in the real world and does not know about the uncertainty structure.
- Is the Nash equilibrium a^* robust to a small amount of payoff uncertainty?

  I.e., Is it "close" to some Bayes Nash equilibrium of any incomplete information game "close" to $\mathbf{g}$?
- Not all equilibria are robust.

  Cf. Email game.
- Sufficient conditions?

Complete Information Games

- Set of players $I = \{1, \dots, |I|\}$
- Action set A_i (finite)
- Payoff function $g_i\colon A \to \mathbb{R}$

  Fix players and actions, and identify the complete information game with $\mathbf{g} = (g_i)_{i \in I}$.
- g_i is extended to $\Delta(A_{-i})$ by

$$g_i(a_i, \lambda_i) = \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \qquad (\lambda_i \in \Delta(A_{-i})).$$

- The set of i's best responses to $\lambda_i \in \Delta(A_{-i})$:

$$br_i(\lambda_i) = \{a_i \in A_i \mid g_i(a_i, \lambda_i) \geq g_i(a_i', \lambda_i)\ \forall\, a_i' \in A_i\}.$$

Correlated Equilibrium and Nash Equilibrium

- Action distribution $\xi \in \Delta(A)$ is an *η-correlated equilibrium* of $\mathbf{g}$ if for all $i \in I$ and all $f_i \colon A_i \to A_i$,

$$\sum_{a \in A} \big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\xi(a) \geq -\eta.$$

- Action distribution $\xi \in \Delta(A)$ is a *correlated equilibrium* of $\mathbf{g}$ if it is a 0-correlated equilibrium of $\mathbf{g}$.
- Action distribution $\xi \in \Delta(A)$ is a Nash equilibrium of $\mathbf{g}$ if it is a correlated equilibrium of $\mathbf{g}$ such that for some $\xi_i \in \Delta(A_i)$, $i \in I$, $\xi(a) = \prod_{i \in I} \xi_i(a_i)$ for all $a \in A$.

$\mathbf{p}$-Dominant Equilibrium

- Let $\mathbf{p} = (p_1, \ldots, p_{|I|}) \in [0, 1]^I$.
- Action profile $a^* \in A$ is a $\mathbf{p}$-*dominant equilibrium* of $\mathbf{g}$ if

  $$a_i^* \in br_i(\lambda_i)$$

  for any $\lambda_i \in \Delta(A_{-i})$ such that $\lambda_i(a_{-i}^*) \geq p_i$.
- Action profile $a^* \in A$ is a *strict* $\mathbf{p}$-*dominant equilibrium* of $\mathbf{g}$ if

  $$\{a_i^*\} = br_i(\lambda_i)$$

  for any $\lambda_i \in \Delta(A_{-i})$ such that $\lambda_i(a_{-i}^*) > p_i$.

Type Spaces

- Type space (T, P):
  - T_i: set of i's types (countable)
  - $P \in \Delta(T)$: common prior

    Assume $P(t_i) = P(\{t_i\} \times T_{-i}) > 0$ for all i and t_i.
- Let

$$P(E_{-i}|t_i) = \frac{P(\{t_i\} \times E_{-i})}{P(t_i)}$$

for $t_i \in T_i$ and $E_{-i} \subset T_{-i}$.

Incomplete Information Games

- Fix I and $(A_i)_{i \in I}$.
- Incomplete information game $(T, P, \mathbf{u})$: $u_i \colon A \times T \to \mathbb{R}$
- i's strategy: $\sigma_i \colon T_i \to \Delta(A_i)$; set of all strategies Σ_i
- $U_i(a_i, \sigma_{-i}|t_i) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) u_i((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i}))$
- The set of i's best responses to σ_{-i}:

  $$BR_i(\sigma_{-i}|t_i) = \{a_i \in A_i \mid U_i(a_i, \sigma_{-i}|t_i) \geq U_i(a_i', \sigma_{-i}|t_i) \, \forall \, a_i' \in A_i\}.$$

- $\sigma \in \Sigma$ is a Bayes Nash equilibrium of $(T, P, \mathbf{u})$ if
  for all $i \in I$, all $a_i \in A_i$, and all $t_i \in T_i$,
  $\sigma_i(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma_{-i}|t_i)$.
- Any $(T, P, \mathbf{u})$ has at least one BNE.
- $\xi \in \Delta(A)$ is an *equilibrium action distribution* of $(T, P, \mathbf{u})$ if
  there exists a BNE σ of $(T, P, \mathbf{u})$ such that
  $\xi(a) = \sum_{t \in T} P(t) \sigma(a|t)$.

Robust Equilibria

- Given $\mathbf{g}$ and $(T, P, \mathbf{u})$, let

$$T_i^{g_i} = \{t_i \in T_i \mid u_i(a, t_i, t_{-i}) = g_i(a) \text{ for all } a \in A \text{ and}$$
$$\text{for all } t_{-i} \in T_{-i} \text{ with } P(t_{-i}|t_i) > 0\},$$

  and $T^{\mathbf{g}} = \prod_{i=1}^{I} T_i^{g_i}$.

- $(T, P, \mathbf{u})$ is an ε-*elaboration* of $\mathbf{g}$ if $P(T^{\mathbf{g}}) = 1 - \varepsilon$.
- $\|\xi - \xi'\| = \max_{a \in A} |\xi(a) - \xi'(a)|$

Definition 1

$\xi \in \Delta(A)$ is *robust to incomplete information* in $\mathbf{g}$ if
for any $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \leq \bar{\varepsilon}$,
any ε-elaboration of $\mathbf{g}$ has an equilibrium action distribution
$\xi' \in \Delta(A)$ such that $\|\xi - \xi'\| \leq \delta$.

Email Game

- A risk-dominated equilibrium is not robust.

  $\because$ For any $\varepsilon > 0$, there exists an ε-elaboration whose Bayes Nash equilibrium is unique and plays the risk-dominant equilibrium with probability 1.

Non-Existence: Example 3.1

Cyclic Matching Pennies Game

- $I = \{1, 2, 3\}$
- $A_1 = A_2 = A_3 = \{H, T, S\}$
- Payoffs:
  - $g_1(S, *, *) = 1$
  - Otherwise, i wants not to match with $i - 1 \pmod 3$:
    $g_1(H, *, H) = -4$, $g_1(H, *, T) = 4$, $g_1(H, *, S) = 0$,
    $g_1(T, *, T) = -4$, $g_1(T, *, H) = 4$, $g_1(T, *, S) = 0$
  - Same for $i = 2, 3$
- Unique NE (strict): (S, S, S)

- $\tilde{\varepsilon} = 1 - \sqrt{1-\varepsilon}$
- $(T, P, \mathbf{u})$:

$$P(t) = \begin{cases} \tilde{\varepsilon}(1-\tilde{\varepsilon})^{3k} & \text{if } t = (k,k,k) \\ \tilde{\varepsilon}(1-\tilde{\varepsilon})^{3k+1} & \text{if } t = (k,k+1,k) \\ \tilde{\varepsilon}(1-\tilde{\varepsilon})^{3k+2} & \text{if } t = (k,k+1,k+1) \\ 0 & \text{otherwise} \end{cases}$$

$(k = 0, 1, \ldots)$

- $T_1^{g_1} = T_1$
- $T_2^{g_2} = T_2 \setminus \{0\}$; for $t_2 = 0$: T is dominant
- $T_3^{g_3} = T_3 \setminus \{0\}$; for $t_3 = 0$: H is dominant
- $P(T^{\mathbf{g}}) = 1 - P(\{(0,0,0),(0,1,0)\}) = 1 - \tilde{\varepsilon} - \tilde{\varepsilon}(1-\tilde{\varepsilon}) = (1-\tilde{\varepsilon})^2$

- Even a unique NE, which is strict, is not robust.
- The induced action distribution is a correlated equilibrium in the limit as $\varepsilon \to 0$.

Correlated Equilibria and ε-Elaborations

Lemma 1

For any $\eta > 0$, there exists $\bar{\varepsilon} > 0$ such that any equilibrium action distribution of any ε-elaboration of $\mathbf{g}$ with $\varepsilon \leq \bar{\varepsilon}$ is an η-correlated equilibrium of $\mathbf{g}$.

Proof

- Take any $\eta > 0$, and let $\bar{\varepsilon} > 0$ be such that $2M\bar{\varepsilon} \leq \eta$, where $M = \max_{i \in I} \max_{a \in A} |g_i(a)|$.
- Let $(T, P, \mathbf{u})$ be any ε-elaboration with $\varepsilon \leq \bar{\varepsilon}$, and let ξ be any equilibrium action distribution of $(T, P, \mathbf{u})$ with the corresponding BNE σ.
- Fix i and $f_i \colon A_i \to A_i$.

- For all $t_i \in T_i^{g_i}$,

$$\sum_{a \in A} \sum_{t_{-i} \in T_{-i}} \big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\sigma(a|t)P(t_{-i}|t_i) \geq 0.$$

  Hence, $\sum_{t_i \in T_i^{g_i}} P(t_i)(\text{LHS}) \geq 0$.

- Decompose
  $\xi(a) = \sum_{t \in T_i^{g_i} \times T_{-i}} \sigma(a|t)P(t) + \sum_{t \in T_i \setminus T_i^{g_i} \times T_{-i}} \sigma(a|t)P(t)$.

- We have

$$\begin{aligned}
&\sum_{a \in A} \big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\xi(a) \\
&\geq -2MP(T_i \setminus T_i^{g_i} \times T_{-i}) \\
&\geq -2M(1 - P(T^{\mathbf{g}})) = -2M\varepsilon \geq -\eta.
\end{aligned}$$

Correlated Equilibria and ε-Elaborations

Lemma 2

Suppose

- $\varepsilon^k \to 0$ *as* $k \to \infty$,
- $(T, P^k, \mathbf{u}^k)$ *is an* ε^k*-elaboration of* $\mathbf{g}$,
- ξ^k *is an equilibrium action distribution of* $(T, P^k, \mathbf{u}^k)$, *and*
- $\xi^k \to \xi$.

Then ξ *is a correlated equilibrium of* $\mathbf{g}$.

Proof

- Fix any i and any f_i.
- First note $\sum_{a\in A}\big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\xi^k(a) \to \sum_{a\in A}\big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\xi(a)$.
- Take any $\eta > 0$.
- For sufficiently large k so that ξ^k is an η-correlated equilibrium of $\mathbf{g}$ (Lemma 1), we have $\sum_{a\in A}\big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\xi(a) \geq \sum_{a\in A}\big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\xi^k(a) - \eta \geq -2\eta$.
- Since $\eta > 0$ has been taken arbitrarily, we have $\sum_{a\in A}\big(g_i(a) - g_i(f_i(a_i), a_{-i})\big)\xi(a) \geq 0$.

Correlated Equilibria and ε-Elaborations

- $\mathcal{E}(\mathbf{g}, \varepsilon)$: set of all ε-elaborations of $\mathbf{g}$
- $N(T, P, \mathbf{u})$ ($\neq \emptyset$): set of all equilibrium action distributions of $(T, P, \mathbf{u})$
- $N(\varepsilon) = \bigcup_{\varepsilon' \leq \varepsilon} \bigcup_{(T,P,\mathbf{u}) \in \mathcal{E}(\mathbf{g},\varepsilon')} N(T, P, \mathbf{u})$
- $N^* = \bigcap_{\varepsilon > 0} \overline{N(\varepsilon)}$

Lemma 3

N^* is equal to the set of correlated equilibria of $\mathbf{g}$.

Unique Correlated Equilibrium

Proposition 4

If $\mathbf{g}$ has a unique correlated equilibrium ξ^*, then ξ^* is the unique robust equilibrium of $\mathbf{g}$.

Proof

- Let ξ^* be the unique correlated equilibrium of $\mathbf{g}$.
- Then $N^* = \{\xi^*\}$ by Lemma 3.
- For any $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that $N(\bar{\varepsilon}) \subset B^\delta(\xi^*)$ (by the compactness of $\Delta(A) \setminus B^\delta(\xi^*)$).

$\mathbf{p}$-Belief Operator

- An event $E \subset T$ is simple if $E = \prod_{i \in I} E_i$ for some $E_i \subset T_i$, $i \in I$.

  Let $\mathcal{S} \subset 2^T$ denote the set of simple events.

- Fix $\mathbf{p} = (p_1, \ldots, p_{|I|})$.

- For $E \in \mathcal{S}$,

$$B_i^{p_i}(E) = \{t_i \in T_i \mid t_i \in E_i \text{ and } P(E_{-i}|t_i) \geq p_i\},$$
$$B_i^{\mathbf{p},n+1}(E) = B_i^{p_i}(\textstyle\prod_{i \in I} B_i^{\mathbf{p},n}(E)),$$
$$C_i^{\mathbf{p}}(E) = \bigcap_{n=1}^{\infty} B_i^{\mathbf{p},n}(E),$$
$$C^{\mathbf{p}}(E) = \prod_{i \in I} C_i^{\mathbf{p}}(E).$$

- $E \in \mathcal{S}$ is $\mathbf{p}$-*evident* if $E_i \subset B_i^{p_i}(E)$ for all $i \in I$.

Proposition 5

For $E \in \mathcal{S}$, $C^{\mathbf{p}}(E)$ is the largest $\mathbf{p}$-evident event contained in E.

Connection to Incomplete Information Games

Proposition 6

Suppose that $a^* \in A$ *is a* $\mathbf{p}$*-dominant equilibrium of* $\mathbf{g}$*.*
Then $(T, P, \mathbf{u})$ *has a BNE* σ *such that* $\sigma(t)(a^*) = 1$
for all $t \in C^{\mathbf{p}}(T^{\mathbf{g}})$*.*

Proof

- Let $\Sigma_i^* \subset \Sigma_i$ be the set of i's strategies σ_i such that $\sigma_i(a_i^*|t_i) = 1$ for all $t_i \in C_i^{\mathbf{p}}(T_i^{g_i})$.
- Define the restricted best response correspondence $\beta_i^* \colon \Sigma_{-i}^* \to \Sigma_i^*$ by

$$\beta_i^*(\sigma_{-i}) = \{\sigma_i' \in \Sigma_i^* \mid \sigma_i'(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma_{-i}|t_i)\}.$$

- Take any $\sigma_{-i} \in \Sigma_{-i}^*$.

  Let $t_i \in C_i^{\mathbf{p}}(T_i^{g_i})$ ($\subset T_i^{g_i}$).

  Since $t_i \in B_i^{p_i}(C_i^{\mathbf{p}}(T_i^{g_i}))$ by the $\mathbf{p}$-evidence of $C_i^{\mathbf{p}}(T_i^{g_i})$, t_i assigns probability at least p_i to the opponents playing a_{-i}^*.
- Therefore, by $\mathbf{p}$-dominance, a_i^* is a best response for t_i.

- This implies that $\beta_i^*(\sigma_{-i}) \neq \emptyset$ for all $\sigma_{-i} \in \Sigma^*_{-i}$.
- Thus, Kakutani's Fixed Point Theorem (with an appropriate topology) guarantees the existence of a BNE of $(T, P, \mathbf{u})$ in Σ^*.

Critical Path Theorem

Theorem 1

For $\mathbf{p} \in [0,1]^I$*, suppose that* $\sum_{i \in I} p_i < 1$*, and let* $\kappa(\mathbf{p}) = (1 - \min_{i \in I} p_i)/(1 - \sum_{i \in I} p_i)$.
Then for any type space (T, P) *and any* $E \in \mathcal{S}$,

$$P(C^{\mathbf{p}}(E)) \geq 1 - \kappa(\mathbf{p})(1 - P(E)).$$

- Generalization with a "simpler and more transparent" proof in Oyama and Takahashi (2020)

Robustness and $\mathbf{p}$-Dominance

Proposition 7

Suppose that $a^* \in A$ is a $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ with $\sum_{i \in I} p_i < 1$.
Then a^* is robust to incomplete information in $\mathbf{g}$.

Proof

- Take any $\delta > 0$, and let $\bar{\varepsilon} = \delta/\kappa(\mathbf{p})$.
- Consider any ε-elaboration $(T, P, \mathbf{u})$ with $\varepsilon \leq \bar{\varepsilon}$.
- By Proposition 6, we can take a BNE σ such that $\sigma(t)(a^*) = 1$ for all $t \in C^{\mathbf{p}}(T^{\mathbf{g}})$.
- By Theorem 1,

$$P(C^{\mathbf{p}}(T^{\mathbf{g}})) \geq 1 - \kappa(p)(1 - P(T^{\mathbf{g}})) = 1 - \kappa(\mathbf{p})\varepsilon.$$

- Therefore, we have

$$\begin{aligned} P(\{t \mid \sigma(t)(a^*) = 1\}) &\geq P(C^{\mathbf{p}}(T^{\mathbf{g}})) \\ &\geq 1 - \kappa(\mathbf{p})\varepsilon \geq 1 - \delta. \end{aligned}$$

Proposition 8

Suppose that $a^* \in A$ is a strict $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ with $\sum_{i \in I} p_i < 1$.
Then a^* is the unique robust equilibrium of $\mathbf{g}$.

Proof

- Let a^* be a strict $\mathbf{p}$-dominant equilibrium of $\mathbf{g}$ with $\sum_{i \in I} p_i \leq 1$.
- Let $q_i = p_i / \sum_{j \in I} p_j \geq p_i$ for each $i \in I$.

  Note that $\sum_{i \in I} q_i = 1$.
- Fix any $\varepsilon > 0$, and consider the following ε-elaboration $(T, P, \mathbf{u})$:

$$P(t) = \begin{cases} \varepsilon(1-\varepsilon)^k q_i & \text{if } t_i = k+1 \text{ and } t_j = k,\ j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_i(a, t) = \begin{cases} g_i(a) & \text{if } t_i \neq 0, \\ 1 & \text{if } t_i = 0 \text{ and } a_i = a_i^*, \\ 0 & \text{if } t_i = 0 \text{ and } a_i \neq a_i^*. \end{cases}$$

- Take any BNE σ of $(T, P, \mathbf{u})$, and show that for all $i \in I$, $\sigma_i(a_i^* | t_i) = 1$ for all $t_i \in T_i$.