Smallest Equilibrium Implementation in Binary-Action Supermodular Games

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Paper

- Morris, S., D. Oyama, and S. Takahashi (2020). "Implementation via Information Design in Binary-Action Supermodular Games."

Base Game

We fix the base game:

- $I = \{1, \ldots, |I|\}$: Set of players
- $A_i = \{0, 1\}$: Binary action set for i $\qquad (A = A_1 \times \cdots \times A_{|I|})$
- Θ: Finite set of states
- $\mu \in \Delta(\Theta)$: Probability distribution over Θ

  Assume full support: $\mu(\theta) > 0$ for all $\theta \in \Theta$
- $u_i \colon A \times \Theta \to \mathbb{R}$: i's payoff function, supermodular:

  $$d_i(a_{-i}, \theta) = u_i(1, a_{-i}, \theta) - u_i(0, a_{-i}, \theta)$$

  increasing in a_{-i}
- Dominance state:

  There exists $\bar{\theta} \in \Theta$ such that $d_i(\mathbf{0}_{-i}, \bar{\theta}) > 0$ for all $i \in I$.

Information Structures

- T_i: Set of types of player i (finite or countably infinite) $(T = T_1 \times \cdots \times T_{|I|})$
- $\pi \in \Delta(T \times \Theta)$: Common prior, consistent with μ:

$$\sum_t \pi(t, \theta) = \mu(\theta)$$

for all $\theta \in \Theta$.

- Together with the base game, an information structure $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ defines an incomplete information game:
  - $\sigma_i \colon T_i \to \Delta(A_i)$: Strategy of player i
  - Bayes Nash equilibrium (BNE) is defined as usual.
  - $E(\mathcal{T})$: Set of BNEs.
  - $\underline{\sigma} = (\underline{\sigma}_i)_{i \in I}$: Smallest (pure-strategy) BNE
- The *outcome* $\nu \in \Delta(A \times \Theta)$ induced by information structure $\mathcal{T}$ and strategy profile σ:

$$\nu(a, \theta) = \sum_t \pi(t, \theta) \prod_{i \in I} \sigma_i(t_i)(a_i).$$

- Outcome ν satisfies *consistency* if $\sum_{a \in A} \nu(a, \theta) = \mu(\theta)$ for all $\theta \in \Theta$.

Partial Implementation

Definition 1

Outcome ν is *partially implementable* if there exist an information structure $\mathcal{T}$ and an equilibrium $\sigma \in E(\mathcal{T})$ that induce ν.

Proposition 1

Outcome ν is partially implementable if and only if it is a Bayes correlated equilibrium.

- BCE: Set of partially implementable outcomes, or equivalently Bayes correlated equilibria

Smallest Equilibrium Implementation (S-Implementation)

Definition 2

Outcome ν is S-implementable if there exists an information structure $\mathcal{T}$ such that $(\mathcal{T}, \underline{\sigma})$ induces ν.

Definition 3

Outcome ν is fully implementable if there exists an information structure $\mathcal{T}$ such that $(\mathcal{T}, \sigma)$ induces ν for all $\sigma \in E(\mathcal{T})$.

- SI: Set of S-implementable outcomes
- FI: Set of fully implementable outcomes
- $FI \subset SI \subset BCE$

Two-Player Two-State Example (Symmetric Payoffs)

- $I = \{1, 2\}$
- $A_1 = A_2 = \{NI, I\}$
- $\Theta = \{B, G\}$, $\mu(B) = \mu(G) = \frac{1}{2}$
- Payoffs:

B	NI	I
NI	0	0
I	-1	$-1+\varepsilon$

G	NI	I
NI	0	0
I	x	$x+\varepsilon$

  - $\varepsilon > 0$ (supermodularity)
  - $0 < x < 1$, $\varepsilon < \frac{1}{2}(1-x)$
- Designer's objective: maximize the number of players who invest.

Optimal BCE

B	NI	I
NI	$\frac{1-x-2\varepsilon}{2(1-\varepsilon)}$	0
I	0	$\frac{x+\varepsilon}{2(1-\varepsilon)}$

G	NI	I
NI	0	0
I	0	$\frac{1}{2}$

- Conditional on receiving signal I, the average game is:

	NI	I
NI	$0, 0$	$0, -\varepsilon$
I	$-\varepsilon, 0$	$0, 0$

- In this direct information structure, "always play NI" is the smallest equilibrium.
- In fact, this outcome (and any outcome close to it) is not S-implementable.

Sequential Obedience

- Γ: Set of sequences of distinct players
- $\Gamma_i \subset \Gamma$: Set of sequences in which i appears
- $a(\gamma) \in A$: Action profile of all players where players that appear γ play action 1
- $a_{-i}(\gamma) \in A_{-i}$: Action profile of opponent players where players that appear before i in γ play action 1
- $\nu_\Gamma \in \Delta(\Gamma \times \Theta)$: "Ordered outcome"
- Ordered outcome ν_Γ induces an outcome $\nu \in \Delta(A \times \Theta)$ by

$$\nu(a, \theta) = \sum_{\gamma \in \Gamma : a(\gamma) = a} \nu_\Gamma(\gamma, \theta).$$

Definition 4

- Ordered outcome ν_Γ satisfies sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \geq 0$$

  for all $i \in I$.

- Outcome ν satisfies sequential obedience if
  there exists an ordered outcome ν_Γ that induces ν and
  satisfies sequential obedience.

Definition 5

- Ordered outcome ν_Γ satisfies strict sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) > 0$$

  for all $i \in I$ such that $\nu_\Gamma(\Gamma_i \times \Theta) > 0$.

- Outcome ν satisfies strict sequential obedience if there exists an ordered outcome ν_Γ that induces ν and satisfies strict sequential obedience.

Characterization

Theorem 1

1. *If $\nu \in SI$, then it satisfies consistency, obedience, and strict sequential obedience.*
2. *If ν with $\nu(\mathbf{1}, \overline{\theta}) > 0$ satisfies consistency, obedience, and strict sequential obedience, then $\nu \in SI$.*

Corollary 2

$\nu \in \overline{SI}$ if and only if it is satisfies consistency, obedience, and sequential obedience.

Necessity of Sequential Obedience

- Suppose that ν is smallest equilibrium implementable.
- Let $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ be an information structure whose smallest equilibrium induces ν.
- Starting with the smallest strategy $\sigma_i^0(t_i) = 0$ for all $i \in I$ and all $t_i \in T_i$, apply sequential best response in the order $1, 2, \dots, |I|$.
  - $\sigma_i^n(t_i) = 1$ if $i \equiv n \pmod{|I|}$ and $\sum_{t_{-i},\theta} \pi((t_i, t_{-i}), \theta) d_i(\sigma_{-i}^{n-1}(t_{-i}), \theta) > 0$,
  - $\sigma_i^n(t_i) = \sigma_i^{n-1}(t_i)$ otherwise.
- By supermodularity, this process converges monotonically to the smallest equilibrium.

- Let
  - $n_i(t_i) = n$ if $\sigma_i^{n-1}(t_i) = 0$ and $\sigma_i^n(t_i) = 1$, and
  - $n_i(t_i) = \infty$ if $\sigma_i^n(t_i) = 0$ for all n.
- $T(\gamma)$: Set of type profiles t such that $n_i(t_i) = \infty$ if $i \notin S(\gamma)$, and $n_{i_\ell}(t_{i_\ell}) < n_{i_m}(t_{i_m})$ if and only if $\ell < m$
- Define

$$\nu_\Gamma(\gamma, \theta) = \sum_{t \in T(\gamma)} \pi(t, \theta).$$

- ν_Γ induces ν:

$\sum_{\gamma \in \Gamma : a(\gamma) = a} \nu_\Gamma(\gamma, \theta) = \sum_{\gamma \in \Gamma : a(\gamma) = a} \sum_{t \in T(\gamma)} \pi(t, \theta) = \sum_{t \in T : \underline{\sigma}(t) = a} \pi(t, \theta) = \nu(a, \theta).$

- For each $t_i \in T_i$ with $n_i(t_i) < \infty$,

$$\sum_{t_{-i},\theta} \pi((t_i, t_{-i}), \theta) d_i(\sigma_{-i}^{n_i(t_i)-1}(t_{-i}), \theta) > 0.$$

- By adding up the inequality over all such t_i, we have

$$\begin{aligned} 0 &< \sum_{t_i : \, n_i(t_i) < \infty} \sum_{t_{-i},\theta} \pi(t, \theta) d_i(\sigma_{-i}^{n_i(t_i)-1}(t_{-i}), \theta) \\ &= \sum_{\gamma \in \Gamma_i, \theta} \sum_{t \in T(\gamma)} \pi(t, \theta) d_i(a_{-i}(\gamma), \theta) \\ &= \sum_{\gamma \in \Gamma_i, \theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \end{aligned}$$

  for any $i \in I$ such that $\nu_\Gamma(\Gamma_i \times \Theta) > 0$.

- Thus, strict sequential obedience is satisfied.

Sufficiency of Sequential Obedience

- Let $\nu_\Gamma \in \Delta(\Gamma \times \Theta)$ satisfy strict sequential obedience.
- We construct an information structure as follows.
  - $T_i = \{1, 2, \dots\} \cup \{\infty\}$
  - By the assumption $\nu(\mathbf{1}, \overline{\theta}) > 0$,
    $\nu_\Gamma(\bar{\gamma}, \overline{\theta}) > 0$ for some sequence $\bar{\gamma}$ of all players.

    Take $\varepsilon > 0$ such that $\varepsilon < \nu_\Gamma(\bar{\gamma}, \overline{\theta})$.
  - m drawn from $\mathbb{Z}_+$ according to the distribution $\eta(1-\eta)^m$, where $0 < \eta \ll \varepsilon$.
  - γ drawn from Γ according to ν_Γ.
  - Player i receives signal t_i given by

$$t_i = \begin{cases} m + (\text{ranking of } i \text{ in } \gamma) & \text{if } \gamma \in \Gamma_i \\ \infty & \text{otherwise.} \end{cases}$$

- To initiate contagion, re-arrange probabilities:
  - Replace $\nu_\Gamma(\bar{\gamma}, \overline{\theta})$ with $\nu_\Gamma(\bar{\gamma}, \overline{\theta}) - \varepsilon$.
  - Allocate $\frac{\varepsilon}{|I|-1}$ to $(t, \overline{\theta})$ such that $1 \leq t_1 = \cdots = t_{|I|} \leq |I| - 1$.
  - Since $\eta \ll \varepsilon$, types $t_i \in \{1, \ldots, |I| - 1\}$ will assign high probability to $\overline{\theta}$.

- Show by induction that action 1 is the unique action surviving iterated deletion of dominated strategies for all types $t_i < \infty$.

- Initialization step:

  If $t_i \in \{1, \ldots, |I| - 1\}$, the player assigns high probability to $\theta = \overline{\theta}$, and by Dominance State, action 1 is a dominant action.

- Induction step:

  For $\tau \geq |I|$, Suppose all types $t_i \leq \tau - 1$ play action 1.

  Then type $t_i = \tau$ knows that all players before him in the realized sequence play action 1, so his payoff to 1 is at least

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \times (\text{constant}) > 0 \quad \text{as } \eta \approx 0,$$

  where the inequality is by strict sequential obedience.

Two-Player Two-State Example (Symmetric Payoffs)

B	NI	I
NI	0	0
I	-1	$-1+\varepsilon$

G	NI	I
NI	0	0
I	x	$x+\varepsilon$

- S-implemetable outcome:

B	NI	I
NI	$\frac{1-x-\varepsilon}{2-\varepsilon}+\delta$	0
I	0	$\frac{2x+\varepsilon}{2(2-\varepsilon)}-\delta$

G	NI	I
NI	0	0
I	0	$\frac{1}{2}$

- The limit as $\delta \to 0$ attains the supremum when the objective is to maximize the expected number of players who invest.

B	NI	I
NI	0	0
I	-1	$-1+\varepsilon$

G	NI	I
NI	0	0
I	x	$x+\varepsilon$

- By symmetry, consider the symmetric ordered outcome:

	B	G
$\emptyset$	$\frac{1}{2}-2p$	0
12	p	$\frac{1}{4}$
21	p	$\frac{1}{4}$

- Determine p such that sequential obedience is satisfied with equality.

- In the information structure constructed in the sufficiency proof, if the players receive $t_i = \infty$, then they know that $\theta = B$ and play NI as a dominant action.

- Conditional on not receiving $t_i = \infty$, the average game as $\delta \to 0$ is:

	NI	I
NI	$0, 0$	$0, -\frac{\varepsilon}{2}$
I	$-\frac{\varepsilon}{2}, 0$	$\frac{\varepsilon}{2}, \frac{\varepsilon}{2}$

  (I, I): risk-dominant (strictly risk-dominant with $\delta > 0$)

- Then signals as in the Email game are sent, using the dominance state $\overline{\theta} = G$ as "crazy types".

Dual Characterization of Sequential Obedience

- Recall:

  $\nu \in \Delta(A \times \Theta)$ satisfies sequential obedience if there exists $\nu_\Gamma \in \Delta(\Gamma \times \Theta)$ that induces ν and satisfies

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \geq 0 \text{ for all } i \in I. \qquad (\sharp\sharp)$$

Proposition 3

ν satisfies sequential obedience if and only if

$$\sum_{a \in A, \theta \in \Theta} \nu(a, \theta) \max_{\gamma : \bar{a}(\gamma) = a} \sum_{i \in S(a)} \lambda_i d_i(a_{-i}(\gamma), \theta) \geq 0$$

$$\text{for all } (\lambda_i)_{i \in I} \geq 0. \qquad (\sharp)$$

Proof

- Fix $\nu \in \Delta(A \times \Theta)$.
- Let $N_\Gamma(\nu) = \{\nu_\Gamma \in \Delta(\Gamma \times \Theta) \mid \sum_{\gamma:\bar{a}(\gamma)=a} \nu_\Gamma(\gamma, \theta) = \nu(a, \theta)\}$ and $\Lambda(\nu) = \{\lambda \in \Delta(I) \mid \sum_{i \in I(\nu)} \lambda_i = 1\}$.

  (Both are convex and compact.)
- For $\nu_\Gamma \in N_\Gamma(\nu)$ and $\lambda \in \Lambda(\nu)$, let

$$\begin{aligned} D(\nu_\Gamma, \lambda) &= \sum_{i \in I} \lambda_i \sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \\ &= \sum_{\gamma \in \Gamma, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) \sum_{i \in S(\gamma)} \lambda_i d_i(a_{-i}(\gamma), \theta) \\ &= \sum_{a \in A, \theta \in \Theta} \sum_{\gamma:\bar{a}(\gamma)=a} \nu_\Gamma(\gamma, \theta) \sum_{i \in S(a)} \lambda_i d_i(a_{-i}(\gamma), \theta). \end{aligned}$$

  (Linear in each of ν_Γ and λ.)

- By the Minimax Theorem, D has a value D^*:

$$\min_{\lambda \in \Lambda(\nu)} \max_{\nu_\Gamma \in N_\Gamma(\nu)} D(\nu_\Gamma, \lambda) = D^* = \max_{\nu_\Gamma \in N_\Gamma(\nu)} \min_{\lambda \in \Lambda(\nu)} D(\nu_\Gamma, \lambda).$$

- ν satisfies sequential obedience
  $\iff \exists\, \nu_\Gamma \in N_\Gamma(\nu)\ \forall\, \lambda \in \Lambda(\nu)\text{: } D(\nu_\Gamma, \lambda) \geq 0$
  $\iff D^* = \max_{\nu_\Gamma \in N_\Gamma(\nu)} \min_{\lambda \in \Lambda(\nu)} D(\nu_\Gamma, \lambda) \geq 0$

- (LHS of ($\sharp$)) $= \max_{\nu_\Gamma \in N_\Gamma(\nu)} D(\nu_\Gamma, \lambda)$ for each $\lambda \in \Lambda(\nu)$

  Hence,

  ($\sharp$) holds $\iff D^* = \min_{\lambda \in \Lambda(\nu)} \max_{\nu_\Gamma \in N_\Gamma(\nu)} D(\nu_\Gamma, \lambda) \geq 0$

Sequential Obedience in Potential Games

- In potential games,

  the dual condition ($\sharp$) (hence sequential obedience) is equivalent to a simpler *coalitional obedience* condition.

Potential Games

Definition 6

The game is a potential game if there exists $\Phi\colon A \times \Theta \to \mathbb{R}$ such that

$$d_i(a_{-i}, \theta) = \Phi(1, a_{-i}, \theta) - \Phi(0, a_{-i}, \theta).$$

- For each $\nu \in \Delta(A \times \Theta)$, we define a potential for that outcome:

  $$\Phi_\nu(a) = \sum_{a', \theta} \nu(a', \theta) \Phi(a \wedge a', \theta)$$

  where $b = a \wedge a'$ is the action profile such that $b_i = 1$ if and only if $a_i = a'_i = 1$.

Potential Games

- For simplicity, we focus on outcomes ν such that $\nu(\{\mathbf{1}\} \times \Theta) > 0$.

Definition 7

Outcome ν satisfies *coalitional obedience* if

$$\Phi_\nu(\mathbf{1}) \geq \Phi_\nu(a)$$

for all $a \in A$.

Proposition 4

In a potential game, an outcome satisfies sequential obedience if and only if it satisfies coalitional obedience.

- Show that coalitional obedience is equivalent to the dual condition (♯) of sequential obedience.

Convex Potential

- Normalize: $\Phi(\mathbf{0}, \theta) = 0$ for all θ.
- Denote $n(a) = |\{i \in I \mid a_i = 1\}|$.

Definition 8

The potential Φ satisfies *convexity* if

$$\Phi(a, \theta) \leq \frac{n(a)}{|I|}\Phi(\mathbf{1}, \theta) \quad \left(= \left(1 - \frac{n(a)}{|I|}\right)\Phi(\mathbf{0}, \theta) + \frac{n(a)}{|I|}\Phi(\mathbf{1}, \theta)\right)$$

for all θ.

- Because of supermodularity, this is automatically satisfied if Φ is symmetric.
- The potential is convex if and only if the game is not too asymmetric.

Investment Game

- $\Theta = \{1, \dots, |\Theta|\}$
- $d_i(a_{-i}, \theta) = R(\theta) + h_{n(a_{-i})+1} - c_i$
  - h_k: increasing in k
  - $R(\theta)$: strictly increasing in θ
  - $R(|\Theta|) + h_1 > c_i$ for all $i \in I$

    Dominant state is satisfied with $\overline{\theta} = |\Theta|$
  - $c_1 \leq c_2 \leq \cdots \leq c_{|I|}$
- This game has a potential:

$$\Phi(a, \theta) = R(\theta)n(a) + \sum_{k=1}^{n(a)} h_k - \sum_{i \in S(a)} c_i.$$

- Φ satisfies convexity if and only if

$$\frac{1}{\ell}\sum_{k=1}^{\ell}(h_k - c_k) \leq \frac{1}{|I|}\sum_{k=1}^{|I|}(h_k - c_k)$$

  for any $\ell = 1, \ldots, |I| - 1$.

- In particular, a sufficient condition for convexity is:

$$h_k - c_k \leq h_{k+1} - c_{k+1}$$

  for any $k = 1, \ldots, |I| - 1$.

Regime Change Game

- $\Theta = \{1, \dots, |\Theta|\}$
- $d_i(a_{-i}, \theta) = \begin{cases} c_i & \text{if } n(a_{-i}) \geq |I| - k(\theta) \\ c_i - 1 & \text{if } n(a_{-i}) < |I| - k(\theta) \end{cases}$
  - $0 < c_i < 1$
  - $k \colon \Theta \to \mathbb{N}$: strictly increasing, $k(1) \geq 1$
  - $k(|\Theta|) = |I|$

    Dominant state is satisfied with $\overline{\theta} = |\Theta|$
- Action 0: to attack the regime
  Action 1: to abstain from attacking
- The regime collapses if $\#(\text{action } 0 \text{ players}) > k(\theta)$

  $\iff \#(\text{action } 1 \text{ players}) < |I| - k(\theta)$

Regime Change Game

- This game has a potential:

$$\Phi(a,\theta) = \begin{cases} \sum_{i \in S(a)} c_i - (|I| - k(\theta)) & \text{if } n(a) \geq |I| - k(\theta), \\ \sum_{i \in S(a)} c_i - n(a) & \text{if } n(a) < |I| - k(\theta). \end{cases}$$

- Φ satisfies convexity if and only if $c_1 = \cdots = c_{|I|}$.

Grand Coalitional Obedience and Perfect Coordination

Definition 9

Outcome ν satisfies *grand coalitional obedience* if

$$\Phi_\nu(\mathbf{1}) \geq \Phi_\nu(\mathbf{0}) = 0,$$

or equivalently,

$$\sum_{a \in A, \theta \in \Theta} \nu(a, \theta)\Phi(a, \theta) \geq 0.$$

Definition 10

Outcome ν satisfies *perfect coordination* if $\nu(a, \theta) > 0$ only for $a \in \{\mathbf{0}, \mathbf{1}\}$.

Proposition 5

Suppose that the potential satisfies convexity.
A perfectly coordinated outcome satisfies sequential obedience if and only if it satisfies grand coalitional obedience.

Information Design with Adversarial Equilibrium Selection

- Information designer's objective function: $V\colon A \times \Theta \to \mathbb{R}$
- $V(a,\theta)$: increasing in a
- Normalization: $V(\mathbf{0},\theta) = 0$ for all θ
- Optimal information design problem with adversarial equilibrium selection:

$$\begin{aligned}&\sup_{\mathcal{T}} \min_{\sigma \in E(\mathcal{T})} \sum_{t \in T, \theta \in \Theta} \pi(t,\theta) V(\sigma(t),\theta)\\ &= \sup_{\mathcal{T}} \sum_{t \in T, \theta \in \Theta} \pi(t,\theta) V(\underline{\sigma}(t),\theta).\end{aligned}$$

- This is equivalent to

$$\sup_{\nu \in SI} \sum_{a \in A, \theta \in \Theta} \nu(a,\theta) V(a,\theta) = \max_{\nu \in \overline{SI}} \sum_{a \in A, \theta \in \Theta} \nu(a,\theta) V(a,\theta).$$

Restricted Convexity

Definition 11

Designer's objective V satisfies *restricted convexity* with respect to potential Φ if

$$V(a,\theta) \leq \frac{n(a)}{|I|} V(\mathbf{1},\theta)$$

whenever $\Phi(a,\theta) > \Phi(\mathbf{1},\theta)$.

Special cases of interest

- Linear preferences

$$V(a,\theta) = n(a)$$

- Full coordination preferences

$$V(a,\theta) = \begin{cases} 1 & \text{if } a = \mathbf{1} \\ 0 & \text{otherwise} \end{cases}$$

- Regime change preferences:
  - Potential

$$\Phi(a,\theta) = \begin{cases} \sum_{i \in S(a)} c_i - (|I| - k(\theta)) & \text{if } n(a) \geq |I| - k(\theta) \\ \sum_{i \in S(a)} c_i - n(a) & \text{if } n(a) < |I| - k(\theta) \end{cases}$$

  - $\Phi(a,\theta) > \Phi(\mathbf{1},\theta)$ holds only when $n(a) < |I| - k(\theta)$.
  - The objective

$$V(a,\theta) = \begin{cases} 1 & \text{if } n(a) \geq |I| - k(\theta) \\ 0 & \text{if } n(a) < |I| - k(\theta) \end{cases}$$

satisfies restricted convexity with respect to Φ.

Perfect Coordination Solution

Theorem 2

Suppose that Φ *satisfies convexity and* V *satisfies restricted convexity with respect to* Φ*.*
Then there exists an optimal outcome of the adversarial information design problem that satisfies perfect coordination.

Proof

- Consider the problem

$$\max_{(\nu(\mathbf{1},\theta))_{\theta\in\Theta}} \sum_{\theta\in\Theta} \nu(\mathbf{1},\theta)V(\mathbf{1},\theta)$$

  with respect to perfect coordination outcomes,

  subject to

  - grand coalitional obedience:

    $\sum_{a\in A,\theta\in\Theta} \nu(a,\theta)\Phi(a,\theta) \geq 0$,
  - consistency:

    $0 \leq \nu(\mathbf{1},\theta) \leq \mu(\theta)$ $(\theta \in \Theta)$.

- Easy to characterize the solution to this problem:
  - Relabel the states as $\Theta = \{1, \ldots, |\Theta|\}$ in such a way that $\frac{\Phi(\mathbf{1},\theta)}{V(\mathbf{1},\theta)}$ is increasing in θ.
  - Ignoring integer issues,

    find θ^* that solves

$$\sum_{\theta > \theta^*} \mu(\theta)\Phi(\mathbf{1}, \theta) = 0 \quad \left(= \sum_{\theta > \theta^*} \mu(\theta)\Phi(\mathbf{0}, \theta) \right).$$

  - Let

$$\nu^*(a, \theta) = \begin{cases} \mu(\theta) & \text{if } a = \mathbf{1} \text{ and } \theta > \theta^*, \\ \mu(\theta) & \text{if } a = \mathbf{0} \text{ and } \theta \leq \theta^*, \\ 0 & \text{otherwise.} \end{cases}$$

- We want to show that ν^* is an optimal outcome of the adversarial information design problem.
- Take any $\nu \in \overline{SI}$.
- Show that there exists a perfect coordination outcome ν' satisfying consistency such that
  - grand coalitional obedience is satisfied (by convexity of Φ), and
  - $\sum_{a,\theta} \nu'(a,\theta)V(a,\theta) \geq \sum_{a,\theta} \nu(a,\theta)V(a,\theta)$ (by restricted convexity of V).

If $\nu(a,\theta) > 0$ for $a \neq \mathbf{0}, \mathbf{1}$, split $\nu(a,\theta)$ to $(\mathbf{0},\theta)$ and $(\mathbf{1},\theta)$ appropriately.