Smallest Equilibrium Implementation in Binary-Action Supermodular Games

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Game Theory I

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Morris, S., D. Oyama, and S. Takahashi (2020). "Implementation via Information Design in Binary-Action Supermodular Games."

Base Game

We fix the base game:

•
$$I = \{1, \dots, |I|\}$$
: Set of players

• $A_i = \{0, 1\}$: Binary action set for i $(A = A_1 \times \cdots \times A_{|I|})$

Θ: Finite set of states

μ ∈ Δ(Θ): Probability distribution over Θ
 Assume full support: μ(θ) > 0 for all θ ∈ Θ

▶ $u_i: A \times \Theta \to \mathbb{R}$: *i*'s payoff function, supermodular:

$$d_i(a_{-i}, \theta) = u_i(1, a_{-i}, \theta) - u_i(0, a_{-i}, \theta)$$

increasing in a_{-i}

Dominance state:

There exists $\overline{\theta} \in \Theta$ such that $d_i(\mathbf{0}_{-i}, \overline{\theta}) > 0$ for all $i \in I$.

Information Structures

- *T_i*: Set of types of player *i* (finite or countably infinite) (*T* = *T*₁ × · · · × *T*_{|*I*|})
- $\pi \in \Delta(T \times \Theta)$: Common prior, consistent with μ :

$$\sum_t \pi(t,\theta) = \mu(\theta)$$

for all $\theta \in \Theta$.

- Together with the base game, an information structure $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ defines an incomplete information game:
 - $\sigma_i \colon T_i \to \Delta(A_i)$: Strategy of player i
 - Bayes Nash equilibrium (BNE) is defined as usual.
 - $E(\mathcal{T})$: Set of BNEs.
 - $\underline{\sigma} = (\underline{\sigma}_i)_{i \in I}$: Smallest (pure-strategy) BNE
- The outcome ν ∈ Δ(A × Θ) induced by information structure *T* and strategy profile σ:

$$\nu(a,\theta) = \sum_{t} \pi(t,\theta) \prod_{i \in I} \sigma_i(t_i)(a_i).$$

• Outcome ν satisfies *consistency* if $\sum_{a \in A} \nu(a, \theta) = \mu(\theta)$ for all $\theta \in \Theta$.

Partial Implementation

Definition 1

Outcome ν is *partially implementable* if there exist an information structure \mathcal{T} and an equilibrium $\sigma \in E(\mathcal{T})$ that induce ν .

Proposition 1

Outcome ν is partially implementable if and only if it is a Bayes correlated equilibrium.

 BCE: Set of partially implementable outcomes, or equivalently Bayes correlated equilibria Smallest Equilibrium Implementation (S-Implementation)

Definition 2

Outcome ν is S-implementable if there exists an information structure \mathcal{T} such that $(\mathcal{T}, \underline{\sigma})$ induces ν .

Definition 3

Outcome ν is fully implementable if there exists an information structure \mathcal{T} such that (\mathcal{T}, σ) induces ν for all $\sigma \in E(\mathcal{T})$.

- ► SI: Set of S-implementable outcomes
- ► *FI*: Set of fully implementable outcomes
- $\blacktriangleright \ FI \subset SI \subset BCE$

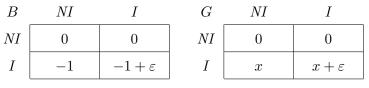
Two-Player Two-State Example (Symmetric Payoffs)

$$I = \{1, 2\}$$

•
$$A_1 = A_2 = \{NI, I\}$$

•
$$\Theta = \{B, G\}, \ \mu(B) = \mu(G) = \frac{1}{2}$$

Payoffs:

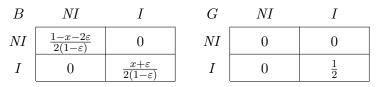


• $\varepsilon > 0$ (supermodularity)

▶ 0 < x < 1, $\varepsilon < \frac{1}{2}(1-x)$

 Designer's objective: maximize the number of players who invest.

Optimal BCE



Conditional on receiving signal I, the average game is:

$$NI \qquad I$$

$$NI \qquad 0,0 \qquad 0,-\varepsilon$$

$$I \qquad -\varepsilon,0 \qquad 0,0$$

- In this direct information structure, "always play NI" is the smallest equilibrium.
- In fact, this outcome (and any outcome close to it) is not S-implementable.

Sequential Obedience

- Γ: Set of sequences of distinct players
- $\Gamma_i \subset \Gamma$: Set of sequences in which *i* appears
- ► a(γ) ∈ A: Action profile of all players where players that appear γ play action 1
- ► a_{-i}(γ) ∈ A_{-i}: Action profile of opponent players where players that appear before i in γ play action 1
- $\nu_{\Gamma} \in \Delta(\Gamma \times \Theta)$: "Ordered outcome"
- ▶ Ordered outcome ν_{Γ} induces an outcome $\nu \in \Delta(A \times \Theta)$ by

$$\nu(a,\theta) = \sum_{\gamma \in \Gamma: a(\gamma) = a} \nu_{\Gamma}(\gamma,\theta).$$

Definition 4

▶ Ordered outcome ν_{Γ} satisfies sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_{\Gamma}(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \ge 0$$

for all $i \in I$.

Outcome ν satisfies sequential obedience if there exists an ordered outcome ν_Γ that induces ν and satisfies sequential obedience.

Definition 5

 \blacktriangleright Ordered outcome ν_{Γ} satisfies strict sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_{\Gamma}(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) > 0$$

for all $i \in I$ such that $\nu_{\Gamma}(\Gamma_i \times \Theta) > 0$.

Outcome ν satisfies strict sequential obedience if there exists an ordered outcome ν_Γ that induces ν and satisfies strict sequential obedience.

Characterization

Theorem 1

- 1. If $\nu \in SI$, then it satisfies consistency, obedience, and strict sequential obedience.
- 2. If ν with $\nu(\mathbf{1}, \overline{\theta}) > 0$ satisfies consistency, obedience, and strict sequential obedience, then $\nu \in SI$.

Corollary 2

 $\nu \in \overline{SI}$ if and only if it is satisfies consistency, obedience, and sequential obedience.

Necessity of Sequential Obedience

- Suppose that ν is smallest equilibrium implementable.
- Let *T* = ((*T_i*)_{*i*∈*I*}, *π*) be an information structure whose smallest equilibrium induces *ν*.
- Starting with the smallest strategy $\sigma_i^0(t_i) = 0$ for all $i \in I$ and all $t_i \in T_i$, apply sequential best response in the order $1, 2, \ldots, |I|$.

$$\begin{array}{l} \blacktriangleright \ \sigma_i^n(t_i) = 1 \ \text{if} \ i \equiv n \ (\text{mod} \ |I|) \ \text{and} \\ \sum_{t_{-i},\theta} \pi((t_i,t_{-i}),\theta) d_i(\sigma_{-i}^{n-1}(t_{-i}),\theta) > 0, \end{array}$$

•
$$\sigma_i^n(t_i) = \sigma_i^{n-1}(t_i)$$
 otherwise.

 By supermodularity, this process converges monotonically to the smallest equilibrium.



•
$$n_i(t_i) = n$$
 if $\sigma_i^{n-1}(t_i) = 0$ and $\sigma_i^n(t_i) = 1$, and
• $n_i(t_i) = \infty$ if $\sigma_i^n(t_i) = 0$ for all n .

• $T(\gamma)$: Set of type profiles t such that $n_i(t_i) = \infty$ if $i \notin S(\gamma)$, and $n_{i_\ell}(t_{i_\ell}) < n_{i_m}(t_{i_m})$ if and only if $\ell < m$

Define

$$\nu_{\Gamma}(\gamma, \theta) = \sum_{t \in T(\gamma)} \pi(t, \theta).$$

 \triangleright ν_{Γ} induces ν :

$$\sum_{\gamma \in \Gamma: a(\gamma) = a} \nu_{\Gamma}(\gamma, \theta) = \sum_{\gamma \in \Gamma: a(\gamma) = a} \sum_{t \in T(\gamma)} \pi(t, \theta) = \sum_{t \in T: \underline{\sigma}(t) = a} \pi(t, \theta) = \nu(a, \theta).$$

For each $t_i \in T_i$ with $n_i(t_i) < \infty$,

$$\sum_{t_{-i},\theta} \pi((t_i, t_{-i}), \theta) d_i(\sigma_{-i}^{n_i(t_i)-1}(t_{-i}), \theta) > 0.$$

• By adding up the inequality over all such t_i , we have

$$0 < \sum_{t_i: n_i(t_i) < \infty} \sum_{t_{-i}, \theta} \pi(t, \theta) d_i(\sigma_{-i}^{n_i(t_i) - 1}(t_{-i}), \theta)$$
$$= \sum_{\gamma \in \Gamma_i, \theta} \sum_{t \in T(\gamma)} \pi(t, \theta) d_i(a_{-i}(\gamma), \theta)$$
$$= \sum_{\gamma \in \Gamma_i, \theta} \nu_{\Gamma}(\gamma, \theta) d_i(a_{-i}(\gamma), \theta)$$

for any $i \in I$ such that $\nu_{\Gamma}(\Gamma_i \times \Theta) > 0$.

Thus, strict sequential obedience is satisfied.

Sufficiency of Sequential Obedience

• Let $\nu_{\Gamma} \in \Delta(\Gamma \times \Theta)$ satisfy strict sequential obedience.

We construct an information structure as follows.

- ► $T_i = \{1, 2, ...\} \cup \{\infty\}$
- By the assumption ν(1, θ) > 0, ν_Γ(γ̄, θ) > 0 for some sequence γ̄ of all players.
 Take ε > 0 such that ε < ν_Γ(γ̄, θ̄).
- *m* drawn from \mathbb{Z}_+ according to the distribution $\eta(1-\eta)^m$, where $0 < \eta \ll \varepsilon$.
- γ drawn from Γ according to ν_{Γ} .
- Player i receives signal t_i given by

$$t_i = \begin{cases} m + (\text{ranking of } i \text{ in } \gamma) & \text{if } \gamma \in \Gamma_i \\ \infty & \text{otherwise.} \end{cases}$$

► To initiate contagion, re-arrange probabilities:

• Replace
$$\nu_{\Gamma}(\bar{\gamma}, \overline{\theta})$$
 with $\nu_{\Gamma}(\bar{\gamma}, \overline{\theta}) - \varepsilon$.

- Allocate $\frac{\varepsilon}{|I|-1}$ to $(t,\overline{\theta})$ such that $1 \leq t_1 = \cdots = t_{|I|} \leq |I| 1$.
- Since $\eta \ll \varepsilon$, types $t_i \in \{1, \ldots, |I| 1\}$ will assign high probability to $\overline{\theta}$.

- Show by induction that action 1 is the unique action surviving iterated deletion of dominated strategies for all types t_i < ∞.</p>
- Initialization step:

If $t_i \in \{1, \ldots, |I| - 1\}$, the player assigns high probability to $\theta = \overline{\theta}$, and by Dominance State, action 1 is a dominant action.

Induction step:

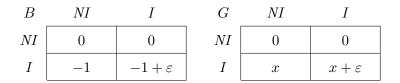
For $\tau \geq |I|$, Suppose all types $t_i \leq \tau - 1$ play action 1.

Then type $t_i = \tau$ knows that all players before him in the realized sequence play action 1, so his payoff to 1 is at least

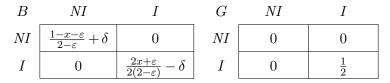
$$\sum_{\gamma\in\Gamma_i,\theta\in\Theta}\nu_{\Gamma}(\gamma,\theta)d_i(a_{-i}(\gamma),\theta)\times(\text{constant})>0\quad\text{as }\eta\approx0,$$

where the inequality is by strict sequential obedience.

Two-Player Two-State Example (Symmetric Payoffs)



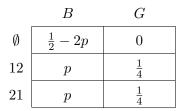
S-implemetable outcome:



▶ The limit as $\delta \rightarrow 0$ attains the supremum when the objective is to maximize the expected number of players who invest.

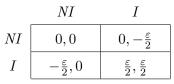
В	NI	Ι	G	NI	Ι
NI	0	0	NI	0	0
Ι	-1	$-1 + \varepsilon$	Ι	x	$x + \varepsilon$

By symmetry, consider the symmetric ordered outcome:



Determine p such that sequential obedience is satisfied with equality.

- In the information structure constructed in the sufficiency proof, if the players receive t_i = ∞, then they know that θ = B and play NI as a dominant action.
- Conditional on not receiving $t_i = \infty$, the average game as $\delta \to 0$ is:



(I,I): risk-dominant (strictly risk-dominant with $\delta > 0$)

Then signals as in the Email game are sent, using the dominance state \(\overline{\theta}\) = G as "crazy types".

Dual Characterization of Sequential Obedience

Recall:

 $\nu\in\Delta(A\times\Theta)$ satisfies sequential obedience if there exists $\nu_{\Gamma}\in\Delta(\Gamma\times\Theta)$ that induces ν and satisfies

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_{\Gamma}(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \ge 0 \text{ for all } i \in I.$$
 (##)

Proposition 3

 ν satisfies sequential obedience if and only if

$$\sum_{a \in A, \theta \in \Theta} \nu(a, \theta) \max_{\gamma: \bar{a}(\gamma) = a} \sum_{i \in S(a)} \lambda_i d_i(a_{-i}(\gamma), \theta) \ge 0$$

for all $(\lambda_i)_{i \in I} \ge 0$. (#)

Proof

Fix
$$\nu \in \Delta(A \times \Theta)$$
.

Let
$$N_{\Gamma}(\nu) = \{\nu_{\Gamma} \in \Delta(\Gamma \times \Theta) \mid \sum_{\gamma:\bar{a}(\gamma)=a} \nu_{\Gamma}(\gamma,\theta) = \nu(a,\theta)\}$$

and $\Lambda(\nu) = \{\lambda \in \Delta(I) \mid \sum_{i \in I(\nu)} \lambda_i = 1\}.$

(Both are convex and compact.)

▶ For $\nu_{\Gamma} \in N_{\Gamma}(\nu)$ and $\lambda \in \Lambda(\nu)$, let

$$\begin{split} D(\nu_{\Gamma},\lambda) &= \sum_{i\in I} \lambda_i \sum_{\gamma\in\Gamma_i,\theta\in\Theta} \nu_{\Gamma}(\gamma,\theta) d_i(a_{-i}(\gamma),\theta) \\ &= \sum_{\gamma\in\Gamma,\theta\in\Theta} \nu_{\Gamma}(\gamma,\theta) \sum_{i\in S(\gamma)} \lambda_i d_i(a_{-i}(\gamma),\theta) \\ &= \sum_{a\in A,\theta\in\Theta} \sum_{\gamma:\bar{a}(\gamma)=a} \nu_{\Gamma}(\gamma,\theta) \sum_{i\in S(a)} \lambda_i d_i(a_{-i}(\gamma),\theta). \end{split}$$

(Linear in each of ν_{Γ} and λ .)

By the Minimax Theorem, D has a value D*:

$$\min_{\lambda \in \Lambda(\nu)} \max_{\nu_{\Gamma} \in N_{\Gamma}(\nu)} D(\nu_{\Gamma}, \lambda) = D^{*} = \max_{\nu_{\Gamma} \in N_{\Gamma}(\nu)} \min_{\lambda \in \Lambda(\nu)} D(\nu_{\Gamma}, \lambda).$$

ν satisfies sequential obedience
⇒ ∃*ν*_Γ ∈ *N*_Γ(*ν*) ∀λ ∈ Λ(*ν*): *D*(*ν*_Γ, λ) ≥ 0
⇒ *D*^{*} = max<sub>*ν*_Γ∈*N*_Γ(*ν*) min_{λ∈Λ(*ν*)} *D*(*ν*_Γ, λ) ≥ 0
(LHS of (\$)) = max<sub>*ν*_Γ∈*N*_Γ(*ν*) *D*(*ν*_Γ, λ) for each λ ∈ Λ(*ν*) Hence,
</sub></sub>

(
$$\sharp$$
) holds $\iff D^* = \min_{\lambda \in \Lambda(\nu)} \max_{\nu_{\Gamma} \in N_{\Gamma}(\nu)} D(\nu_{\Gamma}, \lambda) \ge 0$

Sequential Obedience in Potential Games

In potential games,

the dual condition (#) (hence sequential obedience) is equivalent to a simpler *coalitional obedience* condition.

Potential Games

Definition 6

The game is a potential game if there exists $\Phi\colon A\times\Theta\to\mathbb{R}$ such that

$$d_i(a_{-i},\theta) = \Phi(1,a_{-i},\theta) - \Phi(0,a_{-i},\theta).$$

For each $\nu \in \Delta(A \times \Theta)$, we define a potential for that outcome:

$$\Phi_{\nu}(a) = \sum_{a',\theta} \nu(a',\theta) \Phi(a \wedge a',\theta)$$

where $b=a\wedge a'$ is the action profile such that $b_i=1$ if and only if $a_i=a_i'=1.$

Potential Games

For simplicity, we focus on outcomes ν such that $\nu(\{1\} \times \Theta) > 0.$

Definition 7

Outcome ν satisfies coalitional obedience if

 $\Phi_{\nu}(\mathbf{1}) \geq \Phi_{\nu}(a)$

for all $a \in A$.

Proposition 4

In a potential game, an outcome satisfies sequential obedience if and only if it satisfies coalitional obedience.

Show that coalitional obedience is equivalent to the dual condition (\$\$) of sequential obedience.

Convex Potential

Normalize:
$$\Phi(\mathbf{0}, \theta) = 0$$
 for all θ .

• Denote
$$n(a) = |\{i \in I \mid a_i = 1\}|.$$

Definition 8

The potential Φ satisfies $\mathit{convexity}$ if

$$\Phi(a,\theta) \leq \frac{n(a)}{|I|} \Phi(\mathbf{1},\theta) \quad \left(= \left(1 - \frac{n(a)}{|I|}\right) \Phi(\mathbf{0},\theta) + \frac{n(a)}{|I|} \Phi(\mathbf{1},\theta) \right)$$

for all θ .

- \blacktriangleright Because of supermodularity, this is automatically satisfied if Φ is symmetric.
- The potential is convex if and only if the game is not too asymmetric.

Investment Game

$$\begin{split} \Theta &= \{1, \dots, |\Theta|\} \\ \bullet & d_i(a_{-i}, \theta) = R(\theta) + h_{n(a_{-i})+1} - c_i \\ \bullet & h_k: \text{ increasing in } k \\ \bullet & R(\theta): \text{ strictly increasing in } \theta \\ \bullet & R(|\Theta|) + h_1 > c_i \text{ for all } i \in I \\ \text{ Dominant state is satisfied with } \overline{\theta} = |\Theta| \end{split}$$

$$c_1 \le c_2 \le \cdots \le c_{|I|}$$

This game has a potential:

$$\Phi(a,\theta) = R(\theta)n(a) + \sum_{k=1}^{n(a)} h_k - \sum_{i \in S(a)} c_i.$$

 $\blacktriangleright \ \Phi$ satisfies convexity if and only if

$$\frac{1}{\ell} \sum_{k=1}^{\ell} (h_k - c_k) \le \frac{1}{|I|} \sum_{k=1}^{|I|} (h_k - c_k)$$

for any $\ell = 1, \ldots, |I| - 1$.

In particular, a sufficient condition for convexity is:

$$h_k - c_k \le h_{k+1} - c_{k+1}$$

for any $k = 1, \ldots, |I| - 1$.

Regime Change Game

$$\begin{array}{l} \bullet \ \Theta = \{1, \dots, |\Theta|\} \\ \bullet \ d_i(a_{-i}, \theta) = \begin{cases} c_i & \text{if } n(a_{-i}) \ge |I| - k(\theta) \\ c_i - 1 & \text{if } n(a_{-i}) < |I| - k(\theta) \end{cases} \\ \bullet \ 0 < c_i < 1 \\ \bullet \ k: \Theta \to \mathbb{N}: \text{ strictly increasing, } k(1) \ge 1 \\ \bullet \ k(|\Theta|) = |I| \\ \text{Dominant state is satisfied with } \overline{\theta} = |\Theta| \end{array}$$

- Action 0: to attack the regime Action 1: to abstain from attacking
- ► The regime collapses if #(action 0 players) > k(θ)
 ⇒ #(action 1 players) < |I| − k(θ)</p>

Regime Change Game

This game has a potential:

$$\Phi(a,\theta) = \begin{cases} \sum_{i \in S(a)} c_i - (|I| - k(\theta)) & \text{if } n(a) \ge |I| - k(\theta), \\ \sum_{i \in S(a)} c_i - n(a) & \text{if } n(a) < |I| - k(\theta). \end{cases}$$

• Φ satisfies convexity if and only if $c_1 = \cdots = c_{|I|}$.

Grand Coalitional Obedience and Perfect Coordination

Definition 9

Outcome ν satisfies grand coalitional obedience if

$$\Phi_{\nu}(\mathbf{1}) \ge \Phi_{\nu}(\mathbf{0}) = 0,$$

or equivalently,

$$\sum_{a\in A, \theta\in \Theta}\nu(a,\theta)\Phi(a,\theta)\geq 0.$$

Definition 10

Outcome ν satisfies *perfect coordination* if $\nu(a, \theta) > 0$ only for $a \in \{0, 1\}$.

Proposition 5

Suppose that the potential satisfies convexity. A perfectly coordinated outcome satisfies sequential obedience if and only if it satisfies grand coalitional obedience. Information Design with Adversarial Equilibrium Selection

- ▶ Information designer's objective function: $V \colon A \times \Theta \to \mathbb{R}$
- $V(a, \theta)$: increasing in a
- ▶ Normalization: $V(\mathbf{0}, \theta) = 0$ for all θ
- Optimal information design problem with adversarial equilibrium selection:

$$\begin{split} \sup_{\mathcal{T}} \min_{\sigma \in E(\mathcal{T})} \sum_{t \in T, \theta \in \Theta} \pi(t, \theta) V(\sigma(t), \theta) \\ = \sup_{\mathcal{T}} \sum_{t \in T, \theta \in \Theta} \pi(t, \theta) V(\underline{\sigma}(t), \theta). \end{split}$$

This is equivalent to

$$\sup_{\nu\in SI}\sum_{a\in A,\theta\in\Theta}\nu(a,\theta)V(a,\theta)=\max_{\nu\in\overline{SI}}\sum_{a\in A,\theta\in\Theta}\nu(a,\theta)V(a,\theta).$$

Restricted Convexity

Definition 11

Designer's objective V satisfies restricted convexity with respect to potential Φ if

$$V(a, \theta) \le \frac{n(a)}{|I|} V(\mathbf{1}, \theta)$$

whenever $\Phi(a, \theta) > \Phi(\mathbf{1}, \theta)$.

Special cases of interest

Linear preferences

$$V(a,\theta) = n(a)$$

Full coordination preferences

$$V(a,\theta) = \begin{cases} 1 & \text{if } a = \mathbf{1} \\ 0 & \text{otherwise} \end{cases}$$

Regime change preferences:

Potential

$$\Phi(a,\theta) = \begin{cases} \sum_{i \in S(a)} c_i - (|I| - k(\theta)) & \text{if } n(a) \ge |I| - k(\theta) \\ \sum_{i \in S(a)} c_i - n(a) & \text{if } n(a) < |I| - k(\theta) \end{cases}$$

•
$$\Phi(a,\theta) > \Phi(\mathbf{1},\theta)$$
 holds only when $n(a) < |I| - k(\theta)$.

The objective

$$V(a, \theta) = \begin{cases} 1 & \text{if } n(a) \ge |I| - k(\theta) \\ 0 & \text{if } n(a) < |I| - k(\theta) \end{cases}$$

satisfies restricted convexity with respect to Φ .

Perfect Coordination Solution

Theorem 2

Suppose that Φ satisfies convexity and V satisfies restricted convexity with respect to Φ .

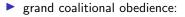
Then there exists an optimal outcome of the adversarial information design problem that satisfies perfect coordination.

Proof

Consider the problem

$$\max_{(\nu(\mathbf{1},\theta))_{\theta\in\Theta}}\sum_{\theta\in\Theta}\nu(\mathbf{1},\theta)V(\mathbf{1},\theta)$$

with respect to perfect coordination outcomes, subject to



$$\sum_{a \in A, \theta \in \Theta} \nu(a, \theta) \Phi(a, \theta) \ge 0$$
,

consistency:

 $0 \leq \nu(\mathbf{1}, \theta) \leq \mu(\theta) \ (\theta \in \Theta).$

• Easy to characterize the solution to this problem:

- ▶ Relabel the states as $\Theta = \{1, ..., |\Theta|\}$ in such a way that $\frac{\Phi(\mathbf{1}, \theta)}{V(\mathbf{1}, \theta)}$ is increasing in θ .
- Ignoring integer issues,

find θ^* that solves

$$\sum_{\theta > \theta^*} \mu(\theta) \Phi(\mathbf{1}, \theta) = 0 \quad \left(= \sum_{\theta > \theta^*} \mu(\theta) \Phi(\mathbf{0}, \theta) \right).$$

Let

$$\nu^*(a,\theta) = \begin{cases} \mu(\theta) & \text{if } a = \mathbf{1} \text{ and } \theta > \theta^*, \\ \mu(\theta) & \text{if } a = \mathbf{0} \text{ and } \theta \le \theta^*, \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that v* is an optimal outcome of the adversarial information design problem.

• Take any
$$\nu \in \overline{SI}$$
.

 \blacktriangleright Show that there exists a perfect coordination outcome ν' satisfying consistency such that

$$\blacktriangleright$$
 grand coalitional obedience is satisfied (by convexity of Φ), and

$$\blacktriangleright \sum_{a,\theta} \nu'(a,\theta) V(a,\theta) \ge \sum_{a,\theta} \nu(a,\theta) V(a,\theta)$$

(by restricted convexity of V).

If $\nu(a,\theta) > 0$ for $a \neq 0, 1$, split $\nu(a,\theta)$ to $(0,\theta)$ and $(1,\theta)$ appropriately.