Full Implementation in Binary-Action Supermodular Games

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Game Theory I

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Paper

- Morris, S., D. Oyama, and S. Takahashi (2020). "Implementation via Information Design in Binary-Action Supermodular Games."

Base Game

- $I = \{1, \ldots, |I|\}$: Set of players
- $A_i = \{0, 1\}$: Binary action set for i $\quad (A = A_1 \times \cdots \times A_{|I|})$
- Θ: Finite set of states
- $\mu \in \Delta(\Theta)$: Probability distribution over Θ

  Assume full support: $\mu(\theta) > 0$ for all $\theta \in \Theta$
- $u_i \colon A \times \Theta \to \mathbb{R}$: i's payoff function, supermodular:

  $$d_i(a_{-i}, \theta) = u_i(1, a_{-i}, \theta) - u_i(0, a_{-i}, \theta)$$

  increasing in a_{-i}
- Two-sided dominance states:

  There exist $\overline{\theta}, \underline{\theta} \in \Theta$ such that $d_i(\mathbf{0}_{-i}, \bar{\theta}) > 0$ and $d_i(\mathbf{1}_{-i}, \underline{\theta}) < 0$ for all $i \in I$.

Information Structures

- T_i: Set of types of player i (finite or countably infinite)
  $(T = T_1 \times \cdots \times T_{|I|})$
- $\pi \in \Delta(T \times \Theta)$: Common prior, consistent with μ:
  $$\sum_t \pi(t, \theta) = \mu(\theta)$$
  for all $\theta \in \Theta$.

- Together with the base game, an information structure $\mathcal{T} = ((T_i)_{i \in I}, \pi)$ defines an incomplete information game:
  - $\sigma_i \colon T_i \to \Delta(A_i)$: Strategy of player i
  - Bayes Nash equilibrium (BNE) is defined as usual.
  - $E(\mathcal{T})$: Set of BNEs.
  - $\underline{\sigma} = (\underline{\sigma}_i)_{i \in I}$: Smallest (pure-strategy) BNE
- The *outcome* $\nu \in \Delta(A \times \Theta)$ induced by information structure $\mathcal{T}$ and strategy profile σ:

$$\nu(a, \theta) = \sum_{t} \pi(t, \theta) \prod_{i \in I} \sigma_i(t_i)(a_i).$$

- Outcome ν satisfies *consistency* if $\sum_{a \in A} \nu(a, \theta) = \mu(\theta)$ for all $\theta \in \Theta$.

Partial Implementation

Definition 1

Outcome ν is *partially implementable* if there exist an information structure $\mathcal{T}$ and an equilibrium $\sigma \in E(\mathcal{T})$ that induce ν.

Proposition 1

Outcome ν is partially implementable if and only if it is a Bayes correlated equilibrium.

- BCE: Set of partially implementable outcomes, or equivalently Bayes correlated equilibria

Smallest Equilibrium Implementation (S-Implementation)

Definition 2
Outcome ν is S-implementable if there exists an information structure $\mathcal{T}$ such that $(\mathcal{T}, \underline{\sigma})$ induces ν.

Definition 3
Outcome ν is fully implementable if there exists an information structure $\mathcal{T}$ such that $(\mathcal{T}, \sigma)$ induces ν for all $\sigma \in E(\mathcal{T})$.

- SI: Set of S-implementable outcomes
- FI: Set of fully implementable outcomes
- $FI \subset SI \subset BCE$

Sequential Obedience

- Γ: Set of sequences of distinct players
- $\Gamma_i \subset \Gamma$: Set of sequences in which i appears
- $a(\gamma) \in A$: Action profile of all players where players that appear γ play action 1
- $a_{-i}(\gamma) \in A_{-i}$: Action profile of opponent players where players that appear before i in γ play action 1
- $\nu_\Gamma \in \Delta(\Gamma \times \Theta)$: "Ordered outcome"
- Ordered outcome ν_Γ induces an outcome $\nu \in \Delta(A \times \Theta)$ by

$$\nu(a, \theta) = \sum_{\gamma \in \Gamma : a(\gamma) = a} \nu_\Gamma(\gamma, \theta).$$

Definition 4

- Ordered outcome ν_Γ satisfies sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) \geq 0$$

  for all $i \in I$.

- Outcome ν satisfies sequential obedience if there exists an ordered outcome ν_Γ that induces ν and satisfies sequential obedience.

Definition 5

- Ordered outcome ν_Γ satisfies strict sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) > 0$$

  for all $i \in I$ such that $\nu_\Gamma(\Gamma_i \times \Theta) > 0$.

- Outcome ν satisfies strict sequential obedience if there exists an ordered outcome ν_Γ that induces ν and satisfies strict sequential obedience.

Reverse Sequential Obedience

- $a^0(\gamma) \in A$: Action profile of all players where players that appear γ play action 0
- $a^0_{-i}(\gamma) \in A_{-i}$: Action profile of opponent players where players that appear before i in γ play action 0
- Ordered outcome $\nu^0_\Gamma \in \Delta(\Gamma \times \Theta)$ *reverse induces* an outcome $\nu \in \Delta(A \times \Theta)$ by

$$\nu(a, \theta) = \sum_{\gamma \in \Gamma : a^0(\gamma) = a} \nu^0_\Gamma(\gamma, \theta).$$

Definition 6

- Ordered outcome ν_Γ^0 satisfies reverse sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma^0(\gamma, \theta) d_i(a_{-i}^0(\gamma), \theta) \leq 0$$

  for all $i \in I$.

- Outcome ν satisfies sequential obedience if
  there exists an ordered outcome ν_Γ^0 that reverse induces ν and satisfies reverse sequential obedience.

Definition 7

- Ordered outcome ν_Γ satisfies strict reverse sequential obedience if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma^0(\gamma, \theta) d_i(a_{-i}^0(\gamma), \theta) < 0$$

  for all $i \in I$ such that $\nu_\Gamma^0(\Gamma_i \times \Theta) > 0$.

- Outcome ν satisfies sequential obedience if there exists an ordered outcome ν_Γ^0 that reverse induces ν and satisfies strict reverse sequential obedience.

Characterization

Theorem 1

1. *If $\nu \in FI$, then it satisfies consistency, strict sequential obedience, and strict reverse sequential obedience.*

2. *If ν with $\nu(\mathbf{1}, \overline{\theta}) > 0$ and $\nu(\mathbf{0}, \underline{\theta}) > 0$ satisfies consistency, strict sequential obedience, and strict reverse sequential obedience, then $\nu \in FI$.*

- Part 1:
  Fully implementable $\Rightarrow$ S-implementable and L-implementable

  Apply the characterization of S-implementability

Part 2

- Let ν satisfy consistency, strict sequential obedience with ν_Γ, and strict reverse sequential obedience with ν_Γ^0.
- Given $\theta \in \Theta$, draw $a \in A$ according to ν;

  and draw $\gamma^+ \in \Pi(S(a))$ and $\gamma^- \in \Pi(I \setminus S(a))$ according to ν_Γ and ν_Γ^0, respectively.
- Draw $m \in \mathbb{Z}_+$ according to $\eta(1-\eta)^m$.
- Each player receives a signal $t_i = (s_i, a_i)$ where

$$s_i = \begin{cases} s_i = m + (\text{ranking of } i \text{ in } \gamma^+) & \text{if } a_i = 1, \\ s_i = m + (\text{ranking of } i \text{ in } \gamma^-) & \text{if } a_i = 0. \end{cases}$$

- Re-arrange probabilities so that the player believes with high probability that the state is $\overline{\theta}$ (resp. $\underline{\theta}$) if $s_i \in \{1, \ldots, |I|\}$ and $\theta = \overline{\theta}$ (resp. $\theta = \underline{\theta}$).

- Then, a similar argument as in the sufficiency proof for S-implementability shows that the player with signal (t_i, a_i) with $a_i = 1$ (resp. $a_i = 0$) plays action 1 (resp. action 0) as a unique rationalizable action.

SO/Reverse SO in Complete Information Games

- $\mathbf{f} = (f_i)_{i \in I}$: Complete information BAS game
- Endow $\Delta(A)$ with the first-order stochastic dominance order.

  For $\xi, \hat{\xi} \in \Delta(A)$, $\hat{\xi}$ first-order stochastically dominates ξ if $\sum_{a \in B} \hat{\xi}(a) \geq \sum_{a \in B} \xi(a)$ for all upper sets $B \subset A$

  (i.e., sets B such that $a' \in B$ whenever $a \in B$ and $a' \geq a$).
- $\overline{X} \subset \Delta(A)$:
  Set of outcomes that satisfy sequential obedience in $\mathbf{f}$

  $\overline{X}^0 \subset \Delta(A)$:
  Set of outcomes that satisfy reverse sequential obedience in $\mathbf{f}$

Proposition 2 (Oyama and Takahashi (2019), Lemma 2(2))

1. $\overline{X}$ *has a largest element, which is degenerate on some action profile* $\overline{a}$.
2. $\overline{X}^0$ *has a smallest element, which is degenerate on some action profile* $\underline{a}$.

Proposition 3 (MOT20, Proposition B.2)

1. $\overline{a}$ *satisfies strict reverse sequential obedience.*
2. $\underline{a}$ *satisfies strict sequential obedience.*

In particular, $\underline{a} \leq \overline{a}$.

- By duality

S-Implementation versus Full Implementation

- Go back to BAS game $(d_i)_{i \in I}$, with the two-sided dominance states assumption.

Lemma 4

If $\nu \in \Delta(A \times \Theta)$ satisfies

- *consistency,*
- *strict sequential obedience, and*
- $\nu(\mathbf{1}, \overline{\theta}) > 0$ *and* $\nu(\mathbf{0}, \underline{\theta}) > 0$,

then there exists $\hat{\nu} \in \Delta(A \times \Theta)$ that

- *first-order stochastically dominates ν and*
- *satisfies consistency,*
- *strict sequential obedience, and*
- *strict reverse sequential obedience, and*
- $\hat{\nu}(\mathbf{1}, \overline{\theta}) > 0$ *and* $\hat{\nu}(\mathbf{0}, \underline{\theta}) > 0$.

Proposition 5

For any $\nu \in \overline{SI}$, *there exists* $\hat{\nu} \in \overline{FI}$ *that first-order stochastically dominates* ν.

- In particular, $FI \neq \emptyset$.

Proposition 6

Suppose that the objective function $V(a, \theta)$ *is nondecreasing in* a. *Then optimal information design with S-implementation is equivalent to that with full implementation.*