Robustness in Binary-Action Supermodular Games

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Game Theory I

October 19, 2023

Papers

- Oyama, D. and S. Takahashi (2020). "Generalized Belief Operator and Robustness in Binary-Action Supermodular Games," Econometrica 88, 693-726.
- Oyama, D. and S. Takahashi (2023). "Robustness in Binary-Action Supermodular Games Revisited."

Complete Information Games

- $I = \{1, \dots, |I|\}$: Set of players
- $A_i = \{0, 1\}$: Action set $\qquad (A = \prod_{i \in I} A_i,\ A_{-i} = \prod_{j \neq i} A_j)$
- $\mathbf{0} = (0, \dots, 0) \in A, \quad \mathbf{1} = (1, \dots, 1) \in A$
- $f_i \colon 2^{I \setminus \{i\}} \to \mathbb{R}$: Payoff gain function
  - $f_i(S)$: i's payoff gain from action 1 over 0 when subset $S \subset I \setminus \{i\}$ of players play action 1
  - Assume supermodularity: $f_i(S)$ increasing in S

We write $\mathbf{f} = (f_i)_{i \in I}$.

Incomplete Information Elaborations

- T_i: (Countable) set of types $\quad (T = \prod_{i \in I} T_i,\ T_{-i} = \prod_{j \neq i} T_j)$
- $P \in \Delta(T)$: Common prior over T
- $u_i \colon A \times T \to \mathbb{R}$: Payoff function $\quad (\mathbf{u} = (u_i)_{i \in I})$

  Write $d_i(S, t) = u_i(\mathbf{1}_{S \cup \{i\}}, t) - u_i(\mathbf{1}_S, t)$.
- Given $\mathbf{f} = (f_i)_{i \in I}$, let

  $$T_i^* = \{t_i \in T_i \mid d_i(S, (t_i, t_{-i})) = f_i(S) \text{ for all } S \in 2^{I \setminus \{i\}} \text{ and for all } t_{-i} \in T_{-i} \text{ with } P(t_{-i}|t_i) > 0\}$$

  $\cdots$ Set of i's types that know that payoffs are given by f_i
- $(T, P, \mathbf{u})$ is an ε-elaboration of $\mathbf{f}$ if

  $$P(T^*) \geq 1 - \varepsilon$$

  i.e., $\Pr(\text{players know that payoffs are given by } \mathbf{f}) \geq 1 - \varepsilon$.

Robustness to Incomplete Information

- $a^* \in A$ is *robust (to incomplete information)* in $\mathbf{f}$ if for any $\delta > 0$, there exists $\varepsilon > 0$ such that for any ε-elaboration of $\mathbf{f}$, there exists an equilibrium $\sigma^* = (\sigma_i^*)_{i \in I}$ such that

$$\sum_{t \in T} P(t) \prod_{i \in I} \sigma_i^*(t_i)(a_i^*) \geq 1 - \delta.$$

- In the following, we study the robustness of $\mathbf{0} = (0, \ldots, 0)$. (OT study that of $\mathbf{1}$.)

Robustness to Canonical Elaborations

- For extreme action profiles (e.g., $\mathbf{0}$) in supermodular games, robustness is equivalent to robustness to "canonical elaborations".
- $(T, P, \mathbf{u})$ is an ε-canonical elaboration of $\mathbf{f}$ if it is an ε-elaboration such that for all $t_i \in T_i \setminus T_i^*$,

  $$d_i(S, (t_i, t_{-i})) = 1 \text{ for all } S \subset I \setminus \{i\} \text{ and all } t_{-i} \in T_{-i}$$

  (and hence action 1 is a dominant action for all $t_i \in T_i \setminus T_i^*$).

- $\mathbf{0}$ is robust if and only if it is *robust to canonical elaborations*, i.e.,

  for any $\delta > 0$, there exists $\varepsilon > 0$ such that
  for any ε-canonical elaboration of $\mathbf{f}$, there exists
  an equilibrium $\sigma^* = (\sigma_i^*)_{i \in I}$ such that

  $$\sum_{t \in T} P(t) \prod_{i \in I} \sigma_i^*(t_i)(0) \geq 1 - \delta.$$

- By supermodularity, this is equivalent to the following:

  for any $\delta > 0$, there exists $\varepsilon > 0$ such that
  for any ε-canonical elaboration of $\mathbf{f}$,

  $$P(\{t \in T \mid \underline{\sigma}_i(t_i)(0) = 1 \text{ for all } i \in I\}) \geq 1 - \delta,$$

  where $\underline{\sigma} = (\underline{\sigma}_i)_{i \in I}$ is the smallest equilibrium.

Result

Theorem 1

For a generic binary-action supermodular game $\mathbf{f}$*,*
the following conditions are equivalent:

1. $\mathbf{0}$ *is robust in* $\mathbf{f}$*.*
2. $\mathbf{0}$ *is a strict monotone potential maximizer in* $\mathbf{f}$*.*
3. *There exists no* $\rho \in \Delta(\Gamma \setminus \{\emptyset\})$ *that satisfies sequential obedience in* $\mathbf{f}$*, i.e.,*

$$\sum_{\gamma \in \Gamma_i} \rho(\gamma) f_i(S_{-i}(\gamma)) \geq 0 \text{ for all } i \in I.$$

- Γ: Set of sequences of distinct players
- $\Gamma_i \subset \Gamma$: Set of sequences in Γ in which i appears
- $S_{-i}(\gamma)$: Set of players that appear before i in γ

- $2 \Rightarrow 1$: By Morris and Ui (2005) for general supermodular games with (finitely) many actions, based on a potential maximization approach.

  Provide an alternative proof based on a higher-order beliefs approach.

- Not $3 \Rightarrow$ not 1: For generic payoffs.

  Show:
  if there exists $\rho \in \Delta(\Gamma \setminus \{\emptyset\})$ that satisfies strict sequential obedience, then for any $\varepsilon > 0$, there exists an ε-elaboration such that $\mathbf{0}$ is never played in the smallest equilibrium.

- $2 \Leftrightarrow 3$: By duality.

Monotone Potential Maximizer (MP-Maximizer)

- $\mathbf{0}$ is a strict MP-maximizer in $\mathbf{f}$ if there exist $v\colon 2^I \to \mathbb{R}$ and $\lambda = (\lambda_i)_{i \in I} \gg 0$ such that

  $$\lambda_i f_i(S) \leq v(S \cup \{i\}) - v(S)$$

  for all $i \in I$ and all $S \subset I \setminus \{i\}$, and $v(\emptyset) > v(S)$ for all $S \neq \emptyset$.

  Such a function v is called a *strict monotone potential* of $\mathbf{f}$ for $\mathbf{0}$.

  - Called "monotone potential maximizer" without "strict" in OT.

Dual Characterization ($2 \Leftrightarrow 3$)

- For a sequence of distinct players $\gamma = (i_1, \ldots, i_k)$, write $S_{-i_\ell}(\gamma) = \{i_1, \ldots, i_{\ell-1}\}$ and $S(\gamma) = \{i_1, \ldots, i_k\}$.
  - Γ: set of all sequences; Γ_i: set of sequences containing i
- There exists a strict monotone potential for $\mathbf{0}$ with weights $\lambda = (\lambda_i)_{i \in I}$ if and only if

$$\sum_{i \in S(\gamma)} \lambda_i f_i(S_{-i}(\gamma)) < 0 \text{ for all } \gamma \in \Gamma \setminus \{\emptyset\}. \qquad (*)$$

- Duality: Either $(*)$ has a solution $\lambda \gg 0$, or

$$\sum_{\gamma \in \Gamma_i} \rho(\gamma) f_i(S_{-i}(\gamma)) \geq 0 \text{ for all } i \in I \qquad (**)$$

  has a solution $\rho \in \Delta(\Gamma \setminus \{\emptyset\})$, but not both.

Proof of “MP-Maximization $\Rightarrow$ Robustness”

- In OT, this is proved as “Generalized Critical Path Theorem”, stated in terms of “generalized belief operator”.
- Here, we prove in terms of best responses.

- Suppose that there exists a strict monotone potential v for $\mathbf{0}$ with weights $(\lambda_i)_{i \in I} \gg 0$:

$$\lambda_i f_i(S) \leq v(S \cup \{i\}) - v(S)$$

  for all $i \in I$ and all $S \subset I \setminus \{i\}$, and $v(\emptyset) > v(S)$ for all $S \neq \emptyset$.
- Fix any ε-canonical elaboration $(T, P, \mathbf{u})$:
  - $d_i(S, (t_i, t_{-i})) = f_i(S)$ for $t_i \in T_i^*$ ("normal types")
  - $P(T^*) \geq 1 - \varepsilon$
  - Action 1: dominant action for $t_i \in T_i \setminus T_i^*$ ("crazy types")

- Starting with the smallest strategy $\sigma_i^0(t_i) = 0$ for all $i \in I$ and all $t_i \in T_i$, apply sequential best response in the order $1, 2, \ldots, |I|$.
- First, let types in $T_i \setminus T_i^*$ switch:

  For $n = 1, \ldots, |I|$,

  - $\sigma_i^n(t_i) = 1$ if $i = n$ and $t_i \in T_i \setminus T_i^*$,
  - $\sigma_i^n(t_i) = \sigma_i^{n-1}(t_i)$ otherwise.
- Then, let types in T_i^* switch:

  For $n = |I| + 1, \ldots$,

  - $\sigma_i^n(t_i) = 1$ if $i \equiv n \pmod{|I|}$ and $\sum_{t_{-i}} P(t_i, t_{-i}) f_i(S(\sigma_{-i}^{n-1}(t_{-i}))) > 0$,
  - $\sigma_i^n(t_i) = \sigma_i^{n-1}(t_i)$ otherwise.
- By supermodularity, this process converges monotonically to the smallest equilibrium.

- Let
  - $n_i(t_i) = n$ if $\sigma_i^{n-1}(t_i) = 0$ and $\sigma_i^n(t_i) = 1$, and
  - $n_i(t_i) = \infty$ if $\sigma_i^n(t_i) = 0$ for all n.

  Write $n(t) = (n_1(t_1), \ldots, n_{|I|}(t_{|I|}))$.

- We want to show:

$$P(\{t \in T \mid n(t) = (\infty, \ldots, \infty)\}) \geq 1 - \kappa \times (1 - P(T^*))$$

  for some constant $\kappa = \kappa(v)$ that depends only on payoffs in $\mathbf{f}$ through monotone potential v (and is independent of the elaboration).

  $\cdots$ "(Generalized) Critical Path Theorem"

  - Then, we have $P(\{t \in T \mid n(t) = (\infty, \ldots, \infty)\}) \to 1$ as $P(T^*) \to 1$ uniformly over all elaborations.

- For $t_i \in T_i^*$ such that $n_i(t_i) < \infty$,

$$\sum_{t_{-i}} P(t_i, t_{-i}) f_i(S(\sigma_{-i}^{n_i(t_i)-1}(t_{-i}))) > 0.$$

- Add these incentive conditions across such t_i's, multiple by $\lambda_i > 0$, and then add across players.
- Notation:
  - For $\gamma = (i_1, \ldots, i_k)$:

    $S(\gamma) = \{i_1, \ldots, i_k\}$

    $T(\gamma)$: Set of type profiles t such that $n_i(t_i) = \infty$ if $i \notin S(\gamma)$, and $n_{i_\ell}(t_{i_\ell}) < n_{i_m}(t_{i_m})$ if and only if $\ell < m$
  - For $t = (t_i)_{i \in I}$:

    $S^*(t) = \{i \in I \mid t_i \in T_i^*\}$

We have

$$
\begin{aligned}
0 &\leq \sum_i \lambda_i \sum_{t_i \in T_i^* : n_i(t_i) < \infty} \sum_{t_{-i}} P(t_i, t_{-i}) f_i(S(\sigma_{-i}^{n_i(t_i)-1}(t_{-i}))) \\
&= \sum_{\gamma \in \Gamma} \sum_{t \in T(\gamma)} P(t) \sum_{i \in S(\gamma) \cap S^*(t)} \lambda_i f_i(S_{-i}(\gamma)) \\
&\leq \sum_{\gamma \in \Gamma} \sum_{t \in T(\gamma)} P(t) \sum_{i \in S(\gamma) \cap S^*(t)} \big(v(S_{-i}(\gamma) \cup \{i\}) - v(S_{-i}(\gamma)) \big) \\
&= \sum_{\gamma \in \Gamma} \sum_{t \in T(\gamma)} P(t) \big(v(S(\gamma)) - v(S(\gamma) \setminus S^*(t)) \big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\gamma\in\Gamma} \sum_{t\in T(\gamma)\cap T^*} P(t)\big(v(S(\gamma)) - v(\emptyset)\big) \\
&\quad + \sum_{\gamma\in\Gamma} \sum_{t\in T(\gamma)\setminus T^*} P(t)\big(v(S(\gamma)) - v(S(\gamma) \setminus S^*(t))\big) \\
&\leq \sum_{\gamma\in\Gamma\setminus\{\emptyset\}} \sum_{t\in T(\gamma)\cap T^*} P(t)(v' - v(\emptyset)) + \sum_{\gamma\in\Gamma} \sum_{t\in T(\gamma)\setminus T^*} P(t)M \\
&= P(\{t \in T^* \mid n(t) \neq (\infty,\dots,\infty)\})(v' - v(\emptyset)) + P(T \setminus T^*)M,
\end{aligned}
$$

where

$$
\begin{aligned}
v' &= \max_{S\neq\emptyset} v(S), \\
M &= \max_{S\supset S'\neq\emptyset} (v(S) - v(S')).
\end{aligned}
$$

- Hence we have

$$P(\{t \in T^* \mid n(t) \neq (\infty, \dots, \infty)\}) \leq \frac{M}{v(\emptyset) - v'} P(T \setminus T^*).$$

- Finally, we have

$$\begin{aligned}
&1 - P(\{t \in T \mid n(t) = (\infty, \dots, \infty)\}) \\
&= P(T \setminus T^*) + P(\{t \in T^* \mid n(t) \neq (\infty, \dots, \infty)\}) \\
&\leq \underbrace{\left(1 + \frac{M}{v(\emptyset) - v'}\right)}_{=\kappa(v)} (1 - P(T^*)).
\end{aligned}$$

$\mathbf{p}$-Dominance (Kajii and Morris 1997)

- For $\mathbf{p} = (p_1, \ldots, p_{|I|}) \in [0,1]^I$, consider the game

$$f_i(S) = \begin{cases} p_i - 1 & \text{if } S = \emptyset, \\ p_i & \text{otherwise.} \end{cases}$$

- $\mathbf{0}$ is a $\mathbf{p}$-dominant equilibrium:

  For all $i \in I$ and all $\alpha_i \in \Delta(2^{I \setminus \{i\}})$ such that $\alpha_i(\emptyset) \geq p_i$, $\sum_{S \subset I \setminus \{i\}} \alpha_i(S) f_i(S) = p_i - \alpha_i(\emptyset) \leq 0$.

- This is a potential game with potential

$$v(S) = \begin{cases} 1 - \sum_{i \in I} p_i & \text{if } S = \emptyset, \\ -\sum_{i \in I \setminus S} p_i & \text{otherwise.} \end{cases}$$

- $v(\emptyset) > v(S)$ for all $S \neq \emptyset$ if and only if $\sum_{i \in I} p_i < 1$.

- For this v,

$$\begin{aligned}\kappa(v) &= 1 + \frac{\max_{S \supset S' \neq \emptyset}(v(S) - v(S'))}{v(\emptyset) - \max_{S \neq \emptyset} v(S)} \\ &= 1 + \frac{\max_{i \in I} \sum_{j \neq i} p_j}{1 - \sum_{i \in I} p_i} \\ &= \frac{1 - \min_{i \in I} p_i}{1 - \sum_{i \in I} p_i} = \kappa^{\mathrm{KM}}(\mathbf{p}).\end{aligned}$$

Proof of "Sequential Obedience $\Rightarrow$ Non-Robustness"

- Suppose that there exists $\rho \in \Delta(\Gamma \setminus \{\emptyset\})$ that satisfies strict sequential obedience:

$$\sum_{\gamma \in \Gamma_i} \rho(\gamma) f_i(S_{-i}(\gamma)) > 0$$

$$\text{for all } i \in I \text{ such that } \rho(\Gamma_i) > 0. \quad (***)$$

- By supermodularity, there exists $\emptyset \neq \bar{S} \subset I$ such that there exists $\rho \in \Delta(\Gamma(\bar{S}))$ that satisfies sequential obedience, where $\Gamma(\bar{S})$ is the set of permutations of players in $\bar{S}$.

  (OT 2019, Appendix A.3)

- For any such $\rho \in \Delta(\Gamma(\bar{S}))$, consider the following elaboration:
  - $T_i = \{1, 2, \ldots\}$ for $i \in \bar{S}$;

    $T_i = \{\infty\}$ for $i \in I \setminus \bar{S}$.
  - m drawn from $\mathbb{Z}_+$ according to the distribution $\eta(1-\eta)^m$, where $\eta \approx 0$;
  - γ drawn from $\Gamma \setminus \{\emptyset\}$ according to ρ;
  - Player i receives signal t_i given by

$$t_i = \begin{cases} m + (\text{ranking of } i \text{ in } \gamma) & \text{if } \gamma \in \Gamma_i \\ \infty & \text{otherwise;} \end{cases}$$

  - $1 \le t_i \le |I| - 1$: action 1 dominant.

- In this elaboration, in any equilibrium, all types t_i of $i \in \bar{S}$ play action 1:
  - If types $t_j < \tau$ play action 1, then approximately the payoff for type $t_i = \tau$ is at least

    $$\sum_{\gamma \in \Gamma_i} \rho(\gamma) f_i(S_{-i}(\gamma)) \times \text{const},$$

    which is positive by $(***)$.
- Hence, $\mathbf{0}$ is not played at any $t \in T$.
- This implies that $\mathbf{0}$ is not robust.

Definitions of MP-Maximizer

- (Simplified version of) the original definition by Morris and Ui (2005):

  $v\colon 2^I \to \mathbb{R}$ is a monotone potential of $\mathbf{f}$ for $\mathbf{0}$ if

  $$\min br_i^{g_i}(\pi_i) \le \max br_i^{v}(\pi_i)$$

  for all $i \in I$ and all $\pi_i \in \Delta(2^{I\setminus\{i\}})$,
  and $v(\emptyset) > v(S)$ for all $S \neq \emptyset$. (MU)

- Strict version by Oyama, Takahashi, and Hofbauer (2008):

  $v\colon 2^I \to \mathbb{R}$ is a strict monotone potential for $\mathbf{f}$ for $\mathbf{0}$ if

  $$\max br_i^{g_i}(\pi_i) \le \max br_i^{v}(\pi_i)$$

  for all $i \in I$ and all $\pi_i \in \Delta(2^{I\setminus\{i\}})$,
  and $v(\emptyset) > v(S)$ for all $S \neq \emptyset$. (OTH)

Equivalence (Strict Version)

- Condition (OTH) is equivalent to our definition:

  There exists $\lambda = (\lambda_i)_{i \in I} \gg 0$ such that

$$\begin{aligned}
&\lambda_i f_i(S) \leq v(S \cup \{i\}) - v(S) \\
&\text{for all } i \in I \text{ and all } S \subset I \setminus \{i\}, \\
&\text{and } v(\emptyset) > v(S) \text{ for all } S \neq \emptyset.
\end{aligned} \tag{1}$$

- Assume condition (OTH).

  Fix any $i \in I$.

- Then there exists no $(\pi_i, \delta_i) \in \mathbb{R}_+^{2^{|I|-1}+1}$ such that

$$\begin{aligned}
&\sum_{S \in 2^{I \setminus \{i\}}} \pi_i(S) f_i(S) \geq 0, \\
&- \sum_{S \in 2^{I \setminus \{i\}}} \pi_i(S)(v(S \cup \{i\}) - v(S)) - \delta_i \geq 0, \\
&- \delta_i < 0.
\end{aligned}$$

- By duality (Farkas' Lemma), there exists $(\lambda_{i,1}, \lambda_{i,2}) \in \mathbb{R}_+^2$ such that

$$\begin{aligned}
&\lambda_{i,1} f_i(S) - \lambda_{i,2}(v(S \cup \{i\}) - v(S)) \leq 0 \text{ for all } S \in 2^{I \setminus \{i\}}, \\
&- \lambda_{i,2} \leq -1.
\end{aligned}$$

- If $\lambda_{i,1} = 0$, then $v(\{i\}) - v(\emptyset) < 0$ would be violated.
- Thus, set $\lambda_i = \lambda_{i,1}/\lambda_{i,2} > 0$.

Equivalence (Weak Version)

- Condition (MU) is equivalent to the following:

  There exists $\lambda = (\lambda_i)_{i \in I} \gg 0$ such that

  $$\lambda_i f_i(S) \leq v(S \cup \{i\}) - v(S)$$
  for all $i \in I$ such that $f_i(I \setminus \{i\}) > 0$ and all $S \subset I \setminus \{i\}$,
  and $v(\emptyset) > v(S)$ for all $S \neq \emptyset$. (2)

- Assume condition (MU).

  Fix any $i \in I$ such that $f_i(I \setminus \{i\}) > 0$.

- Then there exists no $\pi_i \in \Delta(2^{I\setminus\{i\}})$ such that

$$\sum_{S\in 2^{I\setminus\{i\}}} \pi_i(S) f_i(S) > 0,$$

$$-\sum_{S\in 2^{I\setminus\{i\}}} \pi_i(S)(v(S\cup\{i\}) - v(S)) > 0.$$

- By duality (Ville's Theorem), there exists $(\lambda_{i,1}, \lambda_{i,2}) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ such that

$$\lambda_{i,1} f_i(S) - \lambda_{i,2}(v(S\cup\{i\}) - v(S)) \leq 0 \text{ for all } S \in 2^{I\setminus\{i\}}.$$

- If $\lambda_{i,2} = 0$ and thus $\lambda_{i,1} > 0$, then $f_i(I\setminus\{i\}) > 0$ would be violated.
- If $\lambda_{i,1} = 0$, then $v(\{i\}) - v(\emptyset) < 0$ would be violated.
- Thus, set $\lambda_i = \lambda_{i,1}/\lambda_{i,2} > 0$.

- Denote $I^1 = \{i \in I \mid f_i(I \setminus \{i\}) > 0\}$.
- Let $\mathbf{f}_{I^1}(\cdot, \mathbf{0}_{I \setminus I^1})$ be the game with players in I^1 where the players in $I \setminus I^1$ are fixed to play action 0.

Proposition 1

$\mathbf{0}$ *is an MP-maximizer in* $\mathbf{f}$ *if and only if* $\mathbf{0}_{I^1}$ *is a strict MP-maximizer in* $\mathbf{f}_{I^1}(\cdot, \mathbf{0}_{I \setminus I^1})$.

Characterization of Robustness of Extreme Action Profiles

Proposition 2

In any binary-action supermodular game, $\mathbf{0}$ is robust if and only if it is an MP-maximizer.

- "If": by Morris and Ui (2005)

  (For any action profile of supermodular games with (finitely) many actions)

- "Only if": by Oyama and Takahashi (2023)

  Does not hold for non-extreme action profiles.