The Limits of Price Discrimination By Dirk Bergemann, Benjamin Brooks, and Stephen Morris

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Overview

Impact of discriminatory pricing on consumer and producer surplus

Hypothesis

- under monopoly
- third degree price discrimination

Example

- price of a lunch in a public school
- price of a train ticket
- price of a drug or a surgical intervention

Overview

- ▷ No information $\Rightarrow$ monopoly price $\Rightarrow$ A
- ▷ Full information $\Rightarrow$ perfect discrimination $\Rightarrow$ B
- ▷ Forced maximize consumer surplus $\Rightarrow$ C

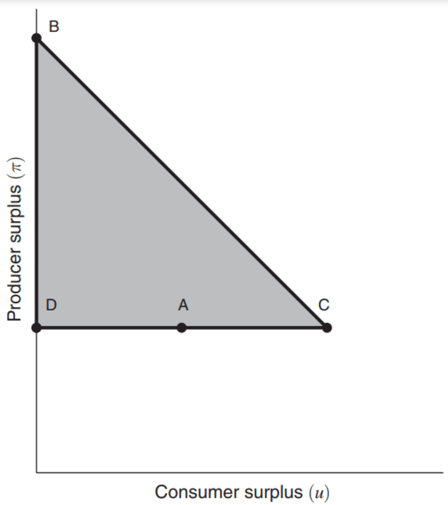

FIGURE 1. THE SURPLUS TRIANGLE

Overview

Remark

- consumer surplus must be non negative
- the producer must get at least the surplus of non information situation
- the sum of consumer and producer surplus cannot exceed the total value limit

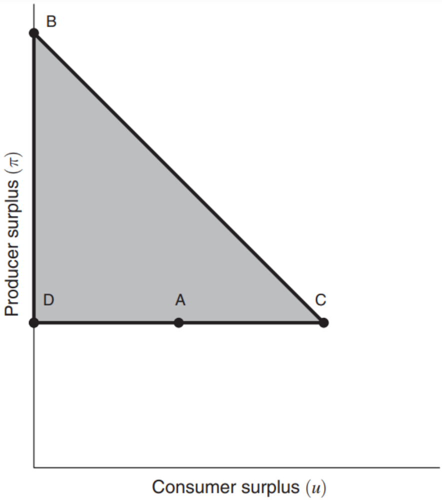

FIGURE 1. THE SURPLUS TRIANGLE

Objective

Construction of a efficient market

- the producer surplus is above the non information situation
- the segmentation of the market should maximise the consumer surplus

Construction

idea : sufficient conditions

With a finite possible prices Let's divide the market into segments, with prices less or equal at the price in uniform monopoly

If :

(i) in each segment, consumers's valuations are always greater that or equal to the price for the segment

(ii) in each segment, the producer is indifferent between charging the price for that segment and charging the uniform monopoly price

Then :

- the producer is indifferent to charging the uniform monopoly price on all segments

i.e. producer surplus must equal uniform monopoly profit

- the allocation is also efficient, so consumers must obtain the rest of the efficient surplus

Iterative construction

- Start with a "lowest price segment" (where a price equal to the lowest valuation will be charged)

-*All consumers with the lowest valuation go into this segment.*

- For each higher valuation, a share of consumers with that valuation also enters into the lowest price segment

-*The relative share of each higher valuation (with respect to each other) is the same as in the prior distribution*

-*The proportion of all of the higher valuations is lower than in the prior distribution*

Iterative Construction

- We can choose that proportion between zero and one such that the producer is indifferent between charging the segment price and the uniform monopoly price
- are in the same relative proportions as they were in the original population
- etc... for the second lowest valuation in the second segment

Setting

- $V = \left\{\nu^1, \ldots, \nu^N\right\}$ with $0 < v_1 < \ldots < v_K$.
- $X \triangleq \Delta(V) = \{x \in \mathbb{R}_+^V \mid \sum_{k=1}^{K} x(\nu_k) = 1\}$: a set of markets
- $x^* \in \Delta(V)$: hold one market as fixed

Setting

- Given a market x, ν_k is optimal if

$$\nu_k \sum_{j=k}^{K} x_j \geq \nu_i \sum_{j=i}^{K} x_j, \forall i = 1, \ldots, K$$

- Let X_k be the set of markets which charging ν_k is optimal, i.e.,

$$X_k \triangleq \left\{ x \in \Delta(V) | \nu_k \sum_{j=k}^{K} x_j \geq v_i \sum_{j=i}^{K} x_j, \forall i = 1, \ldots, K \right\}$$

- Let the maximum feasible surplus as $w^* := \sum_{j=1}^{K} \nu_j x_j^*$
- The uniform price producer surplus is $\pi^* := \max_{k \in 1,\ldots,K} \sum_{j=k}^{K} \nu_k x_j^*$

Three Values with Uniform Probability

- $V = \{1, 2, 3\}$
- $K = 3$ and $\nu_k = k$ and $x^* = \left(\frac{1}{3}; \frac{1}{3}; \frac{1}{3}\right)$
- The feasible social surplus is $w^* = \frac{1}{3}(1 + 2 + 3) = 2$
- The uniform monopoly price is $\nu^* = i^* = 2$
- Under the uniform monopoly price:

$$\pi^* = \frac{2}{3} \times 2 = \frac{4}{3} \qquad u^* = \frac{1}{3}(3 - 2) + \frac{1}{3}(2 - 2) = \frac{1}{3}$$

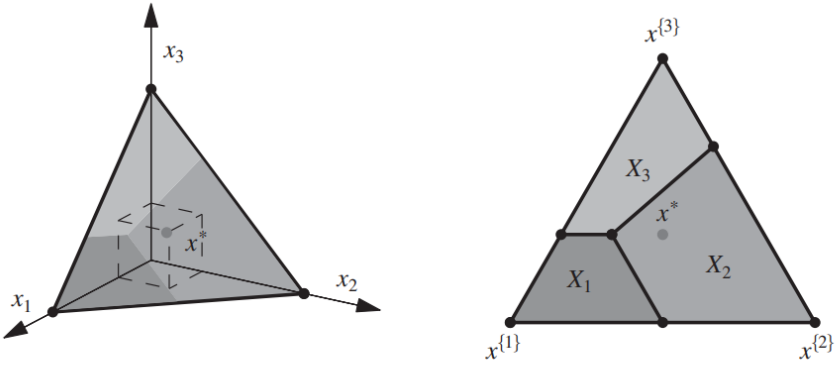

FIGURE 2. THE SIMPLEX OF MARKETS WITH $v_k \in \{1, 2, 3\}$

Example

- $\Sigma = \left\{ \sigma \in \Delta(X) \middle| \sum_{x \in \text{supp}\sigma} x \cdot \sigma(x) = x^* \text{ and } |\text{supp}\,\sigma| < \infty \right\}$: a set of segmentation
- A pricing rule is $\phi : \text{supp}\,\sigma \to \Delta(V)$
- A pricing rule ϕ is optimal if $v_k \in \text{supp}\,\phi(x)$ implies $x \in X_k$.

Segment	x_1	x_2	x_3	$\sigma(x)$	$\text{supp}\phi(x)$
$x^{\{1\}}$	1	0	0	1/3	$\{1\}$
$x^{\{2\}}$	0	1	0	1/3	$\{2\}$
$x^{\{3\}}$	0	0	1	1/3	$\{3\}$
x^*	1/3	1/3	1/3	1	

Given a segmentation σ and pricing rule ψ , consumer surplus is

$$\sum_{x \in supp \sigma} \sigma(x) \sum_{k=1}^{K} \phi(k) \sum_{j=k}^{K} (\nu_j - \nu_k) x_j$$

producer surplus is

$$\sum_{x \in supp \sigma} \sigma(x) \sum_{k=1}^{K} \phi(k) \nu_k \sum_{j=k}^{K} x_j$$

and the total surplus is

$$\sum_{x \in supp \sigma} \sigma(x) \sum_{k=1}^{K} \phi(k) \sum_{j=k}^{K} x_j \nu_j$$

Main Theorem

Now we are ready to state the main theorem formally.

Main theorem

There exists σ and optimal ϕ with consumer surplus u and producer surplus π iff $u \geq 0, \pi \geq \pi^*$, and $u + \pi \leq w^*$.

Only if part is easy. Especially, it is easy to see $u \geq 0$ and $u + \pi \leq w^*$. Since ϕ is optimal, for all $x \in \operatorname{supp} \sigma$,

$$\sum_{k=1}^{K} \phi_k(x) v_i \sum_{j=i}^{K} x_j \geq v_{i^*} \sum_{j=i^*}^{K} x_j$$

where v_{i^*} is uniform monopoly price. Summing up this equations over all $x \in \operatorname{supp} \sigma$,

$$\pi = \sum_{x \in \operatorname{supp} \sigma} \sigma(x) \sum_{k=1}^{K} \phi_k(x) v_i \sum_{j=i}^{K} x_j \geq v_{i^*} \sum_{x \in \operatorname{supp} \sigma} \sigma(x) \sum_{j=i^*}^{K} x_j = \pi^*$$

Main Theorem

Main theorem

There exists σ and ϕ with consumer surplus u and producer surplus π iff $u \geq 0, \pi \geq \pi^*$, and $u + \pi \leq w^*$.

- From now on, we will focus on proving if part. It is easy to achieve $u + \pi = w^*$ or $u = 0$ using these pricing rules.

The maximal and minimal pricing rule

The minimum(maximum) pricing rule is ϕ such that charges $\min(\max) \operatorname{supp} x$ for all x deterministically.

- How can we attain the point between the two? In other words, what happens when u moves between $w^* - \pi$ and 0 with the value π the same? (Note that u and π are linear to σ and ϕ)

Extremal Market

Extremal market

$x \in \Delta(V)$ is extremal market if the producer is indifferent between charging any price in $\operatorname{supp} x$.

Example 1

Under the setting in Example 1, the producer surplus can be $x_1 + x_2 + x_3$, $2x_2 + 2x_3$, or $3x_3$.

Generally, an extremal market x^S with $\operatorname{supp} x = S \subset \{1, \ldots, K\}$ is determined by these $|S| + 1$ equations:

$$v_i \sum_{j=i}^{K} x_i^S = const, \forall i \in S$$

$$\sum_{i \in S} x_i^S = 1$$

Extremal Market

Lemma 1

$X_k = \text{conv}\left(\left\{x^S | k \in S\right\}\right)$

The proof uses two facts.

Krein-Millman Theorem

Let $C \subset \mathbb{R}^n, C \neq \emptyset$, be a compact convex set. Then $C = \text{conv}(\text{ext}(C))$.

Simon 2011, Proposition 15.2

Let $\{l_\alpha\}_{\alpha=1}^m$ be a finite number of linear functionals on $\mathbb{R}^\nu$. Let $\beta_1, \dots, \beta_m \in \mathbb{R}$. Let $K := \cap_{\alpha=1}^m \{x | l_\alpha(x) \geq \beta_\alpha\}$. Let $x \in E(K)$ be an extreme point of K. Then x obeys at least ν distinct equations.

$$l_\alpha(x) = \beta_\alpha$$

Extremal Market

(proof)
Since it is immediate that X_k is convex, $X_k \supset \text{conv}\left(\left\{x^S | k \in S\right\}\right)$.
We are left to prove the converse. By Krein-Millman Theorem, we only have to prove $\left\{x^S | k \in S\right\} = \text{ext}(X^k)$.
Note that $X^k \subset \mathbb{R}^K$ is characterized by these (2K-1) constraints:

$$\sum_{j=1}^{K} x_j = 1$$

$$x_i \geq 0, \forall i \neq k$$

$$v_k \sum_{j=k}^{K} x_j \geq v_i \sum_{j=i}^{K} x_j, \forall i \neq k$$

We can ignore the constraint $x_k \geq 0$ because v_i is optimal $\Rightarrow x_i > 0$

Extremal Market

(proof) By the second fact, all extreme points are characterized by at least K equations out of them. However, we cannot choose $x_i = 0$ and $v_k \sum_{j=k}^{K} x_j = v_i \sum_{j=i}^{K} x_j$ at the same time. Each choice corresponds to x^S. $\square$

Corollary

There exists a segmentation consisting only of extremal markets in X_{i^*}.

Main Theorem

Extremal markets make it easy to move between $u + \pi = w^*$ line and $u = 0$ line.
Suppose that σ consists only of extremal markets. Let us consider ϕ such that charges $\min \operatorname{supp} x$ with probability p and $\max \operatorname{supp} x$ with probability $1 - p$. Then the consumer surplus is:

$$\sum_{x \in \operatorname{supp} \sigma} \sigma(x) \left(p \cdot 0 + (1 - p) \sum_{j=1}^{K} v_j x_j \right)$$

while the producer surplus is the same.

Main Theorem

Now we are ready to prove the main theorem.

Main theorem

There exists σ and ϕ with consumer surplus u and producer surplus π iff $u \geq 0, \pi \geq \pi^*$, and $u + \pi \leq w^*$.

(if part)
By Corollary, there exists a segmentation σ consisting only of extremal markets in X_{i^*}. The maximum and minimum pricing rule under this σ acheive the surplus pairs $(w^* - \pi^*, \pi^*)$ and $(0,\pi^*)$, respectively.
Consider a following segmentation σ':

$$\sigma'(x) = \begin{cases} x_k^* & \text{if } x = x^{\{v_k\}} \\ 0 & \text{o.w.} \end{cases}$$

Charging v_k to market $x^{\{v_k\}}$ under this segmentation achieves the surplus pair $(0,\ w^*)$.

Main Theorem

(if part cont'd)
Note that any surplus pair (u, π) with $u \geq 0, \pi \geq \pi^*$, and $u + \pi \leq w^*$, there exists $\alpha, \beta \in [0, 1]$ such that

$$(u, \pi) = \alpha \cdot (0, w^*) + (1 - \alpha) \cdot [\beta \cdot (w^* - \pi^*, \pi^*) + (1 - \beta) \cdot (0, \pi^*)]$$

The extremal segmentation

$$\sigma''(x) = \alpha\sigma'(x) + (1 - \alpha)\sigma(x)$$

together with the optimal pricing rule that charges $\min \operatorname{supp} x$ with probability β and $\max \operatorname{supp} x$ with probability $1 - \beta$ achieves the desired welfare outcome.(Note that u and π are linear to σ and ϕ)