Robustness and Information Design

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Game Theory I

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Robustness of Equilibrium and Information Design

[OT] Oyama and Takahashi, "Generalized Belief Operator and Robustness in Binary-Action Supermodular Games," *Econometrica*, 2020

[MOT20] Morris, Oyama, and Takahashi, "Implementation via Information Design in Binary-Action Supermodular Games," working paper, 2020 (revised 2023)

[MOT23] Morris, Oyama, and Takahashi, "Strict Robustness to Incomplete Information," *Japanese Economic Review*, 2023

- Robustness (Kajii and Morris 1997)
- Implementation via information design (MOT20)
- Strict robustness (MOT23)

Binary-Action Games

- I: Finite set of players
- Θ: Countable set of states
- $A_i = \{0, 1\}$: Binary action set for player i $\quad (A = \prod_{i \in I} A_i)$

  ($\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$)
- $u_i \colon A \times \Theta \to \mathbb{R}$: player i's payoff (bounded)

  $\mathbf{u} = (u_i)_{i \in I}$

Information Structures (Type Spaces)

- T_i: Countable set of types for player i $\qquad (T = \prod_{i \in I} T_i)$
- $P \in \Delta(T \times \Theta)$: Common prior
- With I and $(A_i)_{i \in I}$ fixed, an incomplete information game is defined by $(\Theta, \mathbf{u}, (T_i)_{i \in I}, P)$.

Robustness (Kajii and Morris 1997)

- $\mathbf{g} = (g_i)_{i \in I}$: Complete information game
- Is a NE a^* of $\mathbf{g}$ approximated by some BNE of any incomplete information game "close" to $\mathbf{g}$?
- An incomplete information game is an ε-elaboration of $\mathbf{g}$ if

$$P(\text{players know that payoffs are } \textbf{equal} \text{ to } \mathbf{g}) \geq 1 - \varepsilon.$$

- $(\Theta, \mathbf{u}, (T_i)_{i \in I}, P)$ is an ε-elaboration of $\mathbf{g}$ if $P(\Theta \times T^{\mathbf{g}}) \geq 1 - \varepsilon$, where

  $$T_i^{g_i} = \{t_i \in T_i \mid u_i(\cdot, \theta) = g_i(\cdot) \text{ for all } \theta \in \Theta \text{ such that } P(\{\theta\} \times T_{-i} | t_i) > 0\}$$

  and $T^{\mathbf{g}} = \prod_{i \in I} T_i^{g_i}$.

- $a^* \in A$ is robust in $\mathbf{g}$ if for any $\delta > 0$, there exists $\varepsilon > 0$ such that any ε-elaboration of $\mathbf{g}$ has a BNE that plays a^* with probability at least $1 - \delta$.

- Not all NE are robust:
  Cf. Email game of Rubinstein (1989).

Sufficient Conditions for Robustness

- Kajii and Morris (1997, Econometrica)
  A $\mathbf{p}$-dominant equilibrium with $\sum_i p_i < 1$ is robust.

  A risk-dominant equilibrium is robust in 2×2 games.
- Ui (2002, Econometrica)
  A potential maximizer is robust in potential games.
- Morris and Ui (2005, JET)
  A monotone potential maximizer (MP-maximizer) is robust if the game or the monotone potential function is supermodular.

Necessity in Binary-Action Supermodular Games (OT)

- OT
  A robust equilibrium must be a monotone potential maximizer in **generic** binary-action supermodular games.
- Proof by contraposition:
  - Suppose that a^* is not an MP-maximizer.
  - . . .
  - Construct an information structure (with "crazy types" with probability ε) in which a^* is never played.

    $\cdots$ Information design!

Implementation via Information Design

- Fix:
  - Finite state space Θ
  - Payoff functions $u_i(a, \theta)$
- What outcomes (i.e., joint distributions over $A \times \Theta$) can be implemented by choosing an information structure?
- Partial implementation:

  An outcome is partially implementable if it is induced by **some** equilibrium of some information structure.
- Well known (Bergemann and Morris 2016):

  An outcome is partially implementable if and only if it satisfies an "obedience" constraint,

  or it is a Bayes correlated equilibrium (BCE).

Full and Smallest Equilibrium Implementation

- An outcome $\nu \in \Delta(A \times \Theta)$ is **fully implementable** if it is induced by all equilibria of some information structure.
- An outcome ν is **smallest equilibrium implementable** if it is induced by the smallest equilibrium of some information structure.
  - Well defined in supermodular games.

    (For each θ, $u_i((a_i', a_{-i}), \theta) - u_i((a_i, a_{-i}), \theta)$ is increasing in a_{-i} where $a_i' > a_i$.)

Characterization in Binary-Action Supermodular Games (MOT20)

- Restrict to $(\Theta, \mathbf{u})$ such that:
  - Finite state space Θ
  - Supermodular payoff functions $u_i(a, \theta)$:

    For each $\theta \in \Theta$, $d_i(a_{-i}, \theta) = u_i((1, a_{-i}), \theta) - u_i((0, a_{-i}), \theta)$ is increasing in a_{-i}.
  - Dominance state:

    There exists $\overline{\theta} \in \Theta$ such that $d_i(\mathbf{0}_{-i}, \overline{\theta}) > 0$ for all i.

- Characterization:

  An outcome is S-implementable if and only if
  it satisfies not only obedience but also **sequential obedience**.

- "Sequential obedience":

  - Designer recommends players to switch to action 1 from action 0 according to a randomly chosen sequence;
  - each player has a strict incentive to switch when told to do so even if he only expects players before him to have switched.

- (An incomplete information generalization of a condition that appeared in an intermediate step in the proof by OT.)

- Full implementation requires "reverse sequential obedience" in addition.

Connection

- Metaphorical interpretation:

  Robustness question can be understood as an information design problem, where an "adversarial" information designer tries to design an information structure such that all BNE are bounded away from a^*.

  - If there is no such information structure, then a^* is robust.
  - Incompatibility between the robustness notion of Kajii and Morris and the information design setting,

    as the former requires that players **know** their payoff functions, whereas no such requirement in the latter.

Strict Robustness (MOT23)

- Robustness against incomplete information perturbations where players believe with $1 - \varepsilon$ that payoff functions are **close** to $\mathbf{g}$.
- A strict MP-maximizer is strictly robust if the game or the strict MP function is supermodular.
- The converse also holds in **all** binary-action supermodular games.

  $\cdots$ By the results on (limit) smallest equilibrium/full implementation by MOT20.

Strict Robustness (MOT23)

- $(\Theta, \mathbf{u}, (T_i)_{i \in I}, P)$ is an (ε, η)-elaboration of $\mathbf{g}$ if $P(T^{\mathbf{g},\eta} \times \Theta) \geq 1 - \varepsilon$, where

$$T_i^{g_i,\eta} = \left\{ t_i \in T_i \;\middle|\; \sum_{\theta \in \Theta} P(\{\theta\} \times T_{-i} | t_i) \max_{a \in A} |u_i(a,\theta) - g_i(a)| \leq \eta \right\}$$

  and $T^{\mathbf{g},\eta} = \prod_{i \in I} T_i^{g_i,\eta}$.

- $a^* \in A$ is strictly robust in $\mathbf{g}$ if for any $\delta > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that any (ε, η)-elaboration of $\mathbf{g}$ has a BNE that plays a^* with probability at least $1 - \delta$.

Robustness (Kajii and Morris 1997)

- $(\Theta, \mathbf{u}, (T_i)_{i \in I}, P)$ is an ε-elaboration of $\mathbf{g}$ if and only if it is an $(\varepsilon, 0)$-elaboration of $\mathbf{g}$.
- Strictly robust $\Rightarrow$ KM-robust
- In a constant payoff game, any action profile is KM-robust, but none is strictly robust.

Limit Smallest Equilibrium Implementation (MOT20)

- Restrict to $(\Theta, \mathbf{u})$ such that:
  - Finite state space Θ
  - Supermodular payoff functions $u_i(a, \theta)$:

    For each $\theta \in \Theta$, $d_i(a_{-i}, \theta) = u_i((1, a_{-i}), \theta) - u_i((0, a_{-i}), \theta)$ is increasing in a_{-i}.
  - Dominance state:

    There exists $\overline{\theta} \in \Theta$ such that $d_i(\mathbf{0}_{-i}, \overline{\theta}) > 0$ for all i.

- $\nu \in \Delta(A \times \Theta)$ is S-implementable in $(\Theta, \mathbf{u})$ if it is induced by the smallest BNE of some information structure.
- $\xi \in \Delta(A)$ is limit S-implementable in $(\Theta, \mathbf{u})$ at θ^* if there exists a sequence $\{\nu^k\}$ of S-implementable outcomes such that $\nu^k(\cdot, \theta^*) \to \xi$.
- Implementing information structures are (ε, η)-elaborations of $\mathbf{u}(\cdot, \theta^*)$ with $\varepsilon, \eta \to 0$.

Equivalence

- $\xi \in \Delta(A)$ is limit S-implementable in $\mathbf{g}$ if there exists $(\Theta, \mathbf{u})$ such that
  - $\mathbf{u}(\cdot, \theta^*) = \mathbf{g}(\cdot)$, and
  - ξ is limit S-implementable in $(\Theta, \mathbf{u})$ at θ^*.
- Focus on $\mathbf{0} \in A$. (Also viewed as an element of $\Delta(A)$.)

Theorem 1

In any BAS game $\mathbf{g}$*, the following are equivalent:*

1. $\mathbf{0}$ *is strictly robust in* $\mathbf{g}$.
2. $\mathbf{0}$ *is the unique action distribution that is limit S-implementable in* $\mathbf{g}$.
3. $\mathbf{0}$ *is the unique action distribution that satisfies* **sequential obedience** *in* $\mathbf{g}$.
4. $\mathbf{0}$ *is a* **strict monotone potential maximizer** *in* $\mathbf{g}$.

- Not 2 $\Rightarrow$ Not 1
- Not 3 $\Rightarrow$ Not 2: by MOT20 (OT)
- 3 $\Leftrightarrow$ 4: by duality (OT)
- 4 $\Rightarrow$ 1: by argument similar to Morris and Ui (2005)

Proof of "Not 3 $\Rightarrow$ Not 2"

- Suppose that $\rho \in \Delta(\Gamma)$ with $\rho(\Gamma \setminus \{\emptyset\}) > 0$ satisfies sequential obedience.

- Define $\nu_\Gamma^k \in \Delta(\Gamma \times \Theta)$ by

$$\nu_\Gamma^k(\gamma, \theta) = \begin{cases} (1 - \frac{1}{k})\rho(\gamma) & \text{if } \theta = \theta^* \\ \frac{1}{k} & \text{if } (\gamma, \theta) = (\bar{\gamma}, \overline{\theta}) \end{cases}$$

  where $\bar{\gamma} \in \Gamma$ is an arbitrarily fixed permutation of all players and $\overline{\theta} \in \Theta$ is the dominance state, where action 1 is strictly dominant for all players.

- Then ν_Γ^k satisfies strict sequential obedience for $\mathbf{u}$:

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma^k(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) > 0$$

  for all $i \in I$.

Non-Extreme Action Profiles

- Assume:

  There exist $\overline{\theta}, \underline{\theta} \in \Theta$ such that $d_i(\mathbf{0}_{-i}, \overline{\theta}) > 0$ and $d_i(\mathbf{1}_{-i}, \underline{\theta}) < 0$ for all $i \in I$.

Theorem 2

In any BAS game $\mathbf{g}$, the following are equivalent:

1. a^* *is strictly robust in* $\mathbf{g}$.
2. a^* *is the unique action distribution that is limit* **fully** *implementable in* $\mathbf{g}$.
3. a^* *is the unique action distribution that satisfies sequential obedience and* **reverse sequential obedience** *in* $\mathbf{g}$.
4. a^* *is a strict monotone potential maximizer in* $\mathbf{g}$.