Mathematics II
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**Homework 3**

Due on May 8

**1.** MWG Exercise 3.B.2.

**2.** Determine whether the following statement is true or false:

If preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is monotone, then it is weakly monotone.

(Prove the statement if it is true; find a counter-example otherwise.)

**3.** Prove the following:

If preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is monotone and continuous, then it is weakly monotone.

**4.** MWG Exercise 3.C.1.

Also determine whether the lexicographic preference relation is quasilinear with respect to each commodity.

**5.** MWG Exercise 3.C.4.

**6.** MWG Exercise 3.C.6.

**7.** Prove part 3 of Proposition 3.8.

**8.** Let $X = \mathbb{R} \times \mathbb{R}^{L-1}_+$, and suppose that $\succsim$ on $X$ is represented by a utility function $u\colon X \to \mathbb{R}$ of the form $u(x) = x_1 + \phi(x_2, \dots, x_L)$. Show that $\succsim$ is convex if and only if the function $\phi$ is concave.

**9.** MWG Exercise 3.D.1.

**10.** MWG Exercise 3.D.2.

**11.** [Advanced] Let $\succsim$ be the lexicographic preference relation on $X \subset \mathbb{R}^2$.

(1) Let $X = \mathbb{Z}^2_+$, where $\mathbb{Z}_+$ is the set of non-negative integers. Is there a utility function that represents $\succsim$?

(2) Let $X = \mathbb{Q}^2_+$, where $\mathbb{Q}_+$ is the set of non-negative rational numbers. Is there a utility function that represents $\succsim$?

Construct it if there is one; give a proof if there is none.

**12.** [Advanced] MWG Exercise 3.C.3.

**13.** [Advanced] Let $X = \mathbb{R} \times \mathbb{R}_+^{L-1}$, and $\succsim$ a complete and transitive preference relation on $X$ that is continuous and quasi-linear with respect to commodity 1.

(1) Prove the following:

For each $z \in \mathbb{R}_+^{L-1}$, there exist $\alpha^0, \alpha^1 \in \mathbb{R}$ such that $\alpha^1 e_1 \succsim (0, z) \succsim \alpha^0 e_1$.

*Hints.* Let

$$U = \{z \in \mathbb{R}_+^{L-1} \mid (0, z) \succsim \alpha e_1 \text{ for all } \alpha \in \mathbb{R}\},$$
$$L = \{z \in \mathbb{R}_+^{L-1} \mid \alpha e_1 \succsim (0, z) \text{ for all } \alpha \in \mathbb{R}\}.$$

Show that $U$ is both closed and open (relative to $\mathbb{R}_+^{L-1}$). Then it must be that $U = \emptyset$ or $U = \mathbb{R}_+^{L-1}$. (A similar argument applies to $L$.)

(2) Then by a similar argument to the proof of Proposition 3.4 under monotonicity, one can show that for each $z \in \mathbb{R}_+^{L-1}$, there exists $\phi(z) \in \mathbb{R}$ such that $(0, z) \sim \phi(z) e_1$.

Prove that the function $\phi$ is continuous.