Mathematics II
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Homework 6

Due on May 29

1. Suppose that a complete and transitive preference relation $\succsim$ on $\mathcal{L}$ satisfies the Independence Axiom.

(a) Show that for any $L, L' \in \mathcal{L}$ and for any $\alpha \in [0, 1]$,

$$L \succsim L' \Longrightarrow L \succsim \alpha L + (1 - \alpha)L' \succsim L'.$$

(b) Show that for any $L^1, \dots, L^K \in \mathcal{L}$ and for any $\alpha_1, \dots, \alpha_K \in \mathbb{R}_+$ such that $\alpha_1 + \dots + \alpha_K = 1$,

$$L^1 \succsim \cdots \succsim L^K \Longrightarrow L^1 \succsim \alpha_1 L^1 + \cdots + \alpha_K L^K \succsim L^K.$$

2. MWG Exercise 6.B.3.

3. MWG Exercise 6.C.3.

(You may use the fact that if u is continuous, then the condition that $u(\frac{1}{2}x + \frac{1}{2}y) \geq \frac{1}{2}u(x) + \frac{1}{2}u(y)$ implies that u is concave.)

4. Prove the following:

$$\lim_{c \to 1} \frac{x^{1-c} - 1}{1 - c} = \log x \quad \text{for all } x > 0.$$

(Do not refer to "l'Hôpital's Theorem"!)

5. MWG Exercise 6.C.20.

6. Do the same exercise as in Exercise 6.C.20 for the lottery that pays $x + \varepsilon x$ with probability 1/2 and $x - \varepsilon x$ with probability 1/2.

7. MWG Exercise 6.D.1.

8. MWG Exercise 6.F.2.

9. Let Ω be the state space (which is assumed to be finite for simplicity). A function $v \colon 2^\Omega \to [0, 1]$ is called a *capacity* if (i) $v(\emptyset) = 0$, (ii) $v(\Omega) = 1$, and (iii) if $E \subset F$, then $v(E) \leq v(F)$. This is an example of "non-additive probability". Note that probability is a special case of capacity which in addition has additivity. For a random variable $X \colon \Omega \to \mathbb{R}$ such that

$$X(\omega) = \begin{cases} x_1 & \text{if } \omega \in E, \\ x_2 & \text{if } \omega \notin E, \end{cases} \qquad x_1 > x_2,$$

the expectation with respect to v is computed as

$$\int X\,dv = x_2 \times v(\Omega) + (x_1 - x_2) \times v(E).$$

Now consider Example 6.F.1 discussed in Exercise 6.F.2. Let W (B) be the event such that a white (black) ball has been picked from urn H (thus, $W \cap B = \emptyset$ and $W \cup B = \Omega$). Let

$$X_W(\omega) = \begin{cases} 1 & \text{if } \omega \in W, \\ 0 & \text{if } \omega \in B, \end{cases} \qquad X_B(\omega) = \begin{cases} 0 & \text{if } \omega \in W, \\ 1 & \text{if } \omega \in B, \end{cases}$$

and for a capacity v,

$$\tilde{U}_W(\text{H}) = \int X_W\,dv, \qquad \tilde{U}_B(\text{H}) = \int X_B\,dv.$$

(a) Find a capacity v for which $U_W(\text{R}) > \tilde{U}_W(\text{H})$ and $U_B(\text{R}) > \tilde{U}_B(\text{H})$.

(b) Given a capacity v, let $\text{core}(v)$ denote the set of probability distributions over Ω such that $p(W) \geq v(W)$ and $p(B) \geq v(B)$. Show that if $v(W) + v(B) \leq 1$, then $\text{core}(v) \neq \emptyset$, and

$$\int X_W\,dv = \min_{p \in \text{core}(v)} \int X_W\,dp, \qquad \int X_B\,dv = \min_{p \in \text{core}(v)} \int X_B\,dp.$$

(Compare Exercise 6.F.2.)