Mathematics II
Daisuke Oyama
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Homework 9

Due on July 10

1. MWG Exercise 17.B.1.

2. MWG Exercise 16.D.1.

3. MWG Exercise 16.D.2.

4. MWG Exercise 16.D.3.

5. Consider a pure exchange economy with L commodities and I consumers, and assume that each consumer i's preference relation $\succsim_i$ on X_i is strongly monotone, where $X_i \subset \mathbb{R}^L_+$ is i's consumption set to be specified in the following.

(1) Let $X_i = \mathbb{Z}^L_+$ for all i, where $\mathbb{Z}_+$ is the set of all nonnegative integers. Does the First Fundamental Theorem of Welfare Economics hold? Give a proof if it holds or a counter-example if it does not.

(2) Let $X_i = \mathbb{R}_+ \times \mathbb{Z}^{L-1}_+$ for all i. Does the First Fundamental Theorem of Welfare Economics hold? Give a proof if it holds or a counter-example if it does not.

6. MWG Exercise 10.B.1.

7. [Advanced] Let $\Omega = \{1, \ldots, L\}$ be the set of states, and $\Delta(\Omega) = \{p \in \mathbb{R}^L_+ \mid \sum_{\ell=1}^L p_\ell = 1\}$ the set of probability distributions over Ω. There are I agents, $1, \ldots, I$. For each $i = 1, \ldots, I$, $P_i \neq \emptyset$ is a convex and closed subset of $\Delta(\Omega)$, which is interpreted as the set of possible prior probability distributions of agent i. We say that the agents share a *common prior* if $\bigcap_{i=1}^I P_i \neq \emptyset$.

$(f_1, \ldots, f_I) \in (\mathbb{R}^L)^I$ is called a *speculative trade* if $\sum_{i=1}^I f_i = 0$, where $f_{i\ell}$ is what agent i receives when state $\ell \in \Omega$ is realized. $(f_1, \ldots, f_I)$ is profitable for agent i if $p_i \cdot f_i > 0$ for all $p_i \in P_i$. It is known that existence of a common prior is equivalent to absence of a speculative trade that is profitable for all agents.

Theorem 1. *A common prior exists if and only if there is no* $(f_1, \ldots, f_I) \in (\mathbb{R}^L)^I$ *such that* $\sum_{i=1}^I f_i = 0$ *and* $p_i \cdot f_i > 0$ *for all* i *and all* $p_i \in P_i$.

We want to prove Theorem 1 by noticing that it is essentially (the combination of) the Fundamental Theorems of Welfare Economics.

For each i, define $u_i \colon \mathbb{R}^L \to \mathbb{R}$ by

(1) $\quad u_i(f_i) = \min_{p_i \in P_i} p_i \cdot f_i.$

Note that $u_i(0) = 0$. We consider the following pure exchange economy: consumer i's preference on $\mathbb{R}^L$ (where the consumption set is $\mathbb{R}^L$) is represented by the utility function u_i, which is continuous and monotone; i's initial endowment is $0 \in \mathbb{R}^L$. $(0, \ldots, 0) \in (\mathbb{R}^L)^I$ is Pareto efficient if there is no $(f_1, \ldots, f_I) \in (\mathbb{R}^L)^I$ such that $\sum_{i=1}^I f_i = 0$ and $u_i(f_i) \geq u_i(0)$ $(= 0)$ for all i with "$>$" for some i; it is weakly Pareto efficient if there is no $(f_1, \ldots, f_I) \in (\mathbb{R}^L)^I$ such that $\sum_{i=1}^I f_i = 0$ and $u_i(f_i) > u_i(0)$ $(= 0)$ for all i. $(p^*, (0, \ldots, 0)) \in \Delta(\Omega) \times (\mathbb{R}^L)^I$ is a Walrasian equilibrium (where $\Delta(\Omega)$ is interpreted as the set of normalized price vectors) if for all i and all $f_i \in \mathbb{R}^L$, $u_i(f_i) > u_i(0)$ $(= 0)$ implies $p^* \cdot f_i > p^* \cdot 0$ $(= 0)$; it is a quasi-equilibrium if for all i and all $f_i \in \mathbb{R}^L$, $u_i(f_i) > u_i(0)$ $(= 0)$ implies $p^* \cdot f_i \geq p^* \cdot 0$ $(= 0)$.

(1) Show that u_i is a concave function.

(2) Show that

$$P_i = \{p_i \in \Delta(\Omega) \mid p_i \cdot f_i \geq u_i(f_i) \text{ for all } f_i \in \mathbb{R}^L\}. \tag{2}$$

(3) Show that if $p^* \in \bigcap_{i=1}^I P_i$, then $(p^*, (0, \ldots, 0))$ is a Walrasian equilibrium.

(4) Show that if $(p^*, (0, \ldots, 0))$ is a quasi-equilibrium, then $p^* \in \bigcap_{i=1}^I P_i$.

(5) Show that there is no $(f_1, \ldots, f_I) \in (\mathbb{R}^L)^I$ such that $\sum_{i=1}^I f_i = 0$ and $p_i \cdot f_i > 0$ for all i and all $p_i \in P_i$ if and only if $(0, \ldots, 0) \in (\mathbb{R}^L)^I$ is weakly Pareto efficient.

(6) Show that Pareto efficiency and weak Pareto efficiency are equivalent in this economy.

(7) Prove the "only if" part of Theorem 1 by arguing that it is the First Welfare Theorem.

(8) Prove the "if" part of Theorem 1 by arguing that it is the Second Welfare Theorem.