1. Preference and Choice

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Microeconomics I

April 17, 2025

Theory of Individual Decision Making

- X: Set of alternatives
- Two approaches:
  - Preference-based
  - Choice-based
- Under certain conditions, these approaches are equivalent.

Preference Relations

- $\succsim$: Preference relation on X
  - $x \succsim y$

    $\cdots$ "x is at least as good as y", "x is weakly preferred to y"
- $\succ$: Strict preference relation, defined by

  $$x \succ y \iff x \succsim y \text{ and } y \not\succsim x$$

  - $x \succ y \cdots$ "x is (strictly) preferred to y"
- $\sim$: Indifference relation, defined by

  $$x \sim y \iff x \succsim y \text{ and } y \succsim x$$

  - $x \sim y \cdots$ "x is indifferent to y"

Definition 1.1

1. $\succsim$ is complete if for all $x, y \in X$, $x \succsim y$ or $y \succsim x$.
2. $\succsim$ is transitive if for all $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$.

- If $\succsim$ is complete and transitive, then it is said to be *rational*.

Proposition 1.1

Suppose that $\succsim$ is complete and transitive.

1. $\succ$ *is irreflexive:* $x \not\succ x$ *for all* $x \in X$.
2. $\succ$ *is transitive: If* $x \succ y$ *and* $y \succ z$, *then* $x \succ z$.
3. $\succ$ *is asymmetric: If* $x \succ y$, *then* $y \not\succ x$.
4. $\sim$ *is reflexive:* $x \sim x$ *for all* $x \in X$.
5. $\sim$ *is transitive: If* $x \sim y$ *and* $y \sim z$, *then* $x \sim z$.
6. $\sim$ *is symmetric: If* $x \sim y$, *then* $y \sim x$.
7. *If* $x \succ y$ *and* $y \succsim z$, *then* $x \succ z$.
   If $x \succsim y$ *and* $y \succ z$, *then* $x \succ z$.

Utility Representation

Definition 1.2

A function $u\colon X \to \mathbb{R}$ is a utility function representing $\succsim$ if for all $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y).$$

- If u represents $\succsim$, then for any strictly increasing function $f\colon \mathbb{R} \to \mathbb{R}$, the function $v\colon X \to \mathbb{R}$ defined by $v(x) = f(u(x))$ also represents $\succsim$.

Proposition 1.2

If $\succsim$ is represented by some utility function, then it is complete and transitive.

Proposition 1.3

Suppose that X is a finite set. If $\succsim$ is complete and transitive, then there exists a utility function representing $\succsim$.

- For $x \in X$, define $L(x) = \{y \in X \mid x \succsim y\}$.
  - For any $x, y \in X$, we have $L(x) \subset L(y)$ of $L(x) \supset L(y)$ by completeness and transitivity. —Why?
- Define $u\colon X \to \mathbb{R}$ by $u(x) = |L(x)|$ (the number of elements of $L(x)$).
- Prove that this function u represents $\succsim$.
- For infinite X, completeness and transitivity are generally not sufficient to guarantee the existence of a utility function (e.g., lexicographic preference relation on $\mathbb{R}^2$).

Choice Rules

- Choice structure $(\mathcal{B}, C)$:
  - $\mathcal{B} \subset 2^X \setminus \{\emptyset\}$: Set of "budget sets"
  - $C\colon \mathcal{B} \to 2^X$ such that $C(B) \subset B$: Choice rule
- For $B \in \mathcal{B}$,
  $C(B) \subset B$ is the set of the alternatives that the DM *might* choose from B, or the set of her *acceptable alternatives* in B.
- Example:
  - $X = \mathbb{R}^L_+$
  - $\mathcal{B}$ consists of the sets $\{x \in X \mid p \cdot x \leq w\}$, $p \in \mathbb{R}^L_{++}$, $w \in \mathbb{R}_{++}$.

WARP

Definition 1.3

$(\mathcal{B}, C)$ satisfies the weak axiom of revealed preference (WARP) if the following condition holds:
For any $x, y \in X$,
if there exists $B \in \mathcal{B}$ such that $x, y \in B$ and $y \in C(B)$,
then for any $B' \in \mathcal{B}$ such that $x, y \in B'$ and $x \in C(B')$, we have $y \in C(B')$.

- Example:

  Suppose $(\mathcal{B}, C)$ satisfies WARP.

  If $C(\{x, y\}) = \{x\}$, then we must have $y \notin C(\{x, y, z\})$.

  Why?

Proposition 1.4

The following statements are equivalent:

- $(\mathcal{B}, C)$ *satisfies WARP.*
- $(\mathcal{B}, C)$ *satisfies the following condition:*

  For any $x, y \in X$ *and any* $B, B' \in \mathcal{B}$,
  if $x, y \in B \cap B'$, $x \in C(B')$, *and* $y \notin C(B')$, *then* $y \notin C(B)$.
- $(\mathcal{B}, C)$ *satisfies the following condition:*

  For any $x, y \in X$ *and any* $B, B' \in \mathcal{B}$,
  if $x, y \in B \cap B'$, $x \in C(B')$, *and* $y \in C(B)$, *then* $y \in C(B')$
  (and $x \in C(B)$*).*

Relationship between the Two Approaches

1. Does a complete and transitive preference relation generate a choice rule that satisfies WARP?
2. Is a choice rule that satisfies WARP rationalized by a complete and transitive preference relation?

Preference-Maximizing Choice Rules

- Given $\succsim$ on X and $\mathcal{B} \subset 2^X \setminus \{\emptyset\}$, define $C^*(\cdot, \succsim) \colon \mathcal{B} \to 2^X$ by

  $$C^*(B, \succsim) = \{x \in B \mid x \succsim y \text{ for all } y \in B\}$$

  for $B \in \mathcal{B}$.

  - $(\mathcal{B}, C^*(\cdot, \succsim))$: Choice structure generated by $\succsim$
  - $C^*(B, \succsim) \neq \emptyset$ if B is a finite set (by completeness and transitivity).

    We need additional conditions to guarantee $C^*(B, \succsim) \neq \emptyset$ for any B.

Proposition 1.5

If $\succsim$ is complete and transitive, then $(\mathcal{B}, C^*(\cdot, \succsim))$ satisfies WARP.

Proof

- Fix any $x, y \in X$, and suppose that there exists $B \in \mathcal{B}$ such that $x, y \in B$ and $y \in C^*(B, \succsim)$.
- This implies that $y \succsim x$.
- Now take any $B' \in \mathcal{B}$ such that $x, y \in B'$ and $x \in C^*(B' \succsim)$.
- Then for all $z \in B'$, since $x \succsim z$, we have $y \succsim z$ by transitivity.
- Therefore, $y \in C^*(B' \succsim)$.
- This means that WARP is satisfied.

Revealed Preference Relations

- Given $(\mathcal{B}, C)$, define $\succsim^*$ on X by

  $$x \succsim^* y \iff \text{there exists } B \in \mathcal{B} \text{ such that } x, y \in B \text{ and } x \in C(B)$$

  for $x, y \in X$.

  - $\succsim^*$: Revealed preference relation

Rationalizability

- Given $\mathcal{B} \subset 2^X \setminus \{\emptyset\}$, $\succsim$ rationalizes C relative to $\mathcal{B}$ if

$$C(B) = \{x \in B \mid x \succsim y \text{ for all } y \in B\} \quad (= C^*(B, \succsim))$$

  for all $B \in \mathcal{B}$.

Proposition 1.6

Suppose that $(\mathcal{B}, C)$ is such that

- *$\mathcal{B}$ includes all nonempty subsets of X of up to three elements;*
- *$C(B) \neq \emptyset$ for all $B \in \mathcal{B}$; and*
- *WARP is satisfied.*

Then $\succsim^*$ is complete and transitive and rationalizes C relative to $\mathcal{B}$ (i.e., $C(B) = C^*(B, \succsim^*)$ for all $B \in \mathcal{B}$).

Proof

- To prove completeness, take any $x, y \in X$.
- By assumption, $\{x, y\} \in \mathcal{B}$ and $C(\{x, y\}) \neq \emptyset$.
- Thus, $x \in C(\{x, y\})$ or $y \in C(\{x, y\})$, i.e., $x \succsim^* y$ or $y \succsim^* x$.
- To prove transitivity,
  take any $x, y, z \in X$, and suppose that $x \succsim^* y$ and $y \succsim^* z$,
  i.e., $x, y \in B$ and $x \in C(B)$ for some $B \in \mathcal{B}$ and $y, z \in B'$
  and $z \in C(B')$ for some $B' \in \mathcal{B}$.
- By assumption, $\{x, y, z\} \in \mathcal{B}$ and $C(\{x, y, z\}) \neq \emptyset$.
- We want to show that $x \in C(\{x, y, z\})$.
  - If $y \in C(\{x, y, z\})$, then $x \in C(\{x, y, z\})$ by WARP.
  - If $z \in C(\{x, y, z\})$, then $y \in C(\{x, y, z\})$ by WARP,
    which implies $x \in C(\{x, y, z\})$ by WARP as above.

- To prove rationalizability, take any $B \in \mathcal{B}$.
- Recall:
  $x \in C^*(B, \succsim^*)$
  $\iff x \in B$ and $\forall\, y \in B : x \succsim^* y$
  $\iff x \in B$ and $\forall\, y \in B \;\exists\, B' \in \mathcal{B} : x, y \in B'$ and $x \in C(B')$
- $C(B) \subset C^*(B, \succsim^*)$:
  - Let $x \in C(B)$, where $x \in B$.
  - For any $y \in B$, we have $x \succsim^* y$ (since $x, y \in B$ and $x \in C(B)$).
  - Hence, $x \in C^*(B, \succsim^*)$.

- $C^*(B, \succsim^*) \subset C(B)$:
  - Let $x \in C^*(B, \succsim^*)$, where $x \in B$.
  - By assumption, $C(B) \neq \emptyset$, i.e., there is some $y \in C(B)$ $(\subset B)$.

    Since $x \in C^*(B, \succsim^*)$ and $y \in B$, we have $x \succsim^* y$, i.e., there exists some $B' \in \mathcal{B}$ such that $x, y \in B'$ and $x \in C(B')$.
  - Then by WARP, we must have $x \in C(B)$.

Consumer Choice

- $X = \mathbb{R}^L_+$
- $\mathcal{B}$ consists of the sets $\{x \in X \mid p \cdot x \leq w\}$, $p \in \mathbb{R}^L_{++}$, $w \in \mathbb{R}_{++}$.
- This $\mathcal{B}$ does not satisfy the assumption that "$\mathcal{B}$ includes all nonempty subsets of X of up to three elements".
- WARP alone does not guarantee the existence of a complete and transitive preference relation that rationalizes the choice rule.