2. Consumer Choice

Daisuke Oyama

Microeconomics I

April 24, 2025

Consumption

- Commodities $\ell = 1, \dots, L$
- Consumption set $X \subset \mathbb{R}^L$

  In most cases, we assume $X = \mathbb{R}^L_+$.
- $x = \begin{pmatrix} x_1 \\ \vdots \\ x_L \end{pmatrix} \in X$: consumption vector/bundle

Budget Sets

- $p = \begin{pmatrix} p_1 \\ \vdots \\ p_L \end{pmatrix} \in \mathbb{R}^L$: price vector

  In most cases, we assume $p \in \mathbb{R}^L_{++}$, or $p \gg 0$ (i.e., $p_\ell > 0$ for all $\ell = 1, \dots, L$).

- $w \in \mathbb{R}$: wealth level

  We assume $w > 0$.

- Budget sets:

  $$B_{p,w} = \{x \in X \mid p \cdot x \leq w\} \qquad (p \gg 0,\ w > 0)$$

Two Approaches

- This chapter: choice-based
  - $\mathcal{B} = \{B_{p,w} \mid p \gg 0, w > 0\}$
  - Choice rule is given
- Next chapter: preference-based
  - $\succsim$ on X
  - Choice rule is derived by preference (or utility) maximization

Demand Correspondences/Functions

- Choice rule: $x(p, w) \subset B_{p,w}$

  $\cdots$ Walrasian demand correspondence

- If $x(p, w)$ is a singleton set for all (p, w), we refer to it as a Walrasian demand *function*.

Definition 2.1

Demand correspondence $x(p, w)$ is homogeneous of degree zero if for any $p \gg 0$ and $w > 0$, we have $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$.

- For any $\alpha > 0$, $B_{\alpha p, \alpha w} = B_{p,w}$.

Definition 2.2

Demand correspondence $x(p, w)$ satisfies Walras' law if for any $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p, w)$.

Comparative Statics

- In the following, we assume that $x(p, w)$ is a function and is differentiable.
- Wealth effects:

$$D_w x(p, w) = \begin{pmatrix} \frac{\partial x_1}{\partial x}(p, w) \\ \vdots \\ \frac{\partial x_L}{\partial x}(p, w) \end{pmatrix} \in \mathbb{R}^{L \times 1}$$

- Price effects:

$$D_p x(p, w) = \begin{pmatrix} \frac{\partial x_1}{\partial p_1}(p, w) & \cdots & \frac{\partial x_1}{\partial p_L}(p, w) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L}{\partial p_1}(p, w) & \cdots & \frac{\partial x_L}{\partial p_L}(p, w) \end{pmatrix} \in \mathbb{R}^{L \times L}$$

Implication of Homogeneity

Proposition 2.1

If the demand function $x(p, w)$ is homogeneous of degree zero, then for any $p \gg 0$ and $w > 0$,

$$\sum_{k=1}^{L} \frac{\partial x_\ell}{\partial p_k}(p, w) p_k + \frac{\partial x_\ell}{\partial w}(p, w) w = 0 \qquad (\ell = 1, \ldots, L),$$

or in matrix notation,

$$D_p x(p, w) p + D_w x(p, w) w = 0.$$

Implications of Walras' Law

Proposition 2.2

If the demand function $x(p, w)$ satisfies Walras' law, then for any $p \gg 0$ and $w > 0$,

$$x_k(p, w) + \sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell}{\partial p_k}(p, w) = 0 \qquad (k = 1, \ldots, L),$$

or in matrix notation,

$$x(p, w)^{\mathrm{T}} + p^{\mathrm{T}} D_p x(p, w) = 0^{\mathrm{T}}.$$

- "A^{T}" denotes the transpose of a matrix A.

Implications of Walras' Law

Proposition 2.3

If the demand function $x(p, w)$ satisfies Walras' law, then for any $p \gg 0$ and $w > 0$,

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell}{\partial w}(p, w) = 1,$$

or in matrix notation,

$$p^{\mathrm{T}} D_w x(p, w) = 1.$$

Weak Axiom of Revealed Preference (WARP)

- Recall:

  $(\mathcal{B}, C)$ satisfies WARP if and only if the following holds:

  For any $x, y \in X$ and any $B, B' \in \mathcal{B}$,
  if $x, y \in B \cap B'$, $x \in C(B')$, and $y \in C(B)$, then $y \in C(B')$ (and $x \in C(B)$).

- Suppose that the choice rule $x(p, w)$ is singleton-valued.

  Then WARP is translated into:

  For any (p, w) and (p', w'), if $x(p', w') \in B_{p,w}$ and $x(p, w) \in B_{p',w'}$, then $x(p, w) = x(p', w')$.

Definition 2.3

The demand function $x(p, w)$ satisfies the WARP if the following condition holds:
For any (p, w) and (p', w'), if $p' \cdot x(p, w) \leq w'$ and $x(p, w) \neq x(p', w')$, then $p \cdot x(p', w') > w$.

Proposition 2.4

Suppose that the demand function $x(p, w)$ *satisfies Walras' law. Then* $x(p, w)$ *satisfies the WARP if and only if the following condition holds:*
For any (p, w) *and* (p', w'), *if* $p' \cdot x(p, w) = w'$ *and* $x(p, w) \neq x(p', w')$, *then* $p \cdot x(p', w') > w$.

Proof

- Assume:

  For any (p, w) and (p', w'),

  $$p' \cdot x(p, w) = w',\ p \cdot x(p', w') \leq w \Rightarrow x(p, w) = x(p', w').$$

  Call this condition "WARP*".

- We want to show that WARP holds:

  For any (p, w) and (p', w'),

  $$p' \cdot x(p, w) \leq w',\ p \cdot x(p', w') \leq w \Rightarrow x(p, w) = x(p', w').$$

- Fix any (p, w) and (p', w'), and write $x = x(p, w)$ and $x' = x(p', w')$.

  Suppose that $p' \cdot x \leq w'$ and $p \cdot x' \leq w$ hold.

- We show that under Walras' law and WARP*, we cannot have $p' \cdot x < w'$ (and we cannot have $p \cdot x' < w$),

  i.e., we must have $p' \cdot x = w'$ (and $p \cdot x' = w$).

  - Then WARP* implies that $x = x'$.

- Suppose that $p' \cdot x < w'$ holds.

- If $p \cdot x' = w$, then by WARP*, we would have $x = x'$ and $p' \cdot x' < w'$, which contradicts Walras' law.

- Therefore, we have $p \cdot x' < w$.

- Let $\alpha \in (0, 1)$ be such that

$$(\alpha p + (1 - \alpha)p') \cdot x = (\alpha p + (1 - \alpha)p') \cdot x'.$$

  (I.e., let $\alpha = \frac{p' \cdot x' - p' \cdot x}{(p \cdot x - p \cdot x') + (p' \cdot x' - p' \cdot x)} \in (0, 1)$.)

  ($p \cdot x = w$ and $p' \cdot x' = w'$ by Walras' law.)

- Let

$$p'' = \alpha p + (1-\alpha)p',$$
$$w'' = p'' \cdot x \quad (= p'' \cdot x').$$

- Write $x'' = x(p'', w'')$.
- If $p \cdot x'' \geq w$ and $p' \cdot x'' \geq w'$, we would have

$$\begin{aligned} p'' \cdot x'' &= (\alpha p + (1-\alpha)p') \cdot x'' \\ &\geq \alpha w + (1-\alpha)w' \\ &> \alpha(p \cdot x) + (1-\alpha)(p' \cdot x) \\ &\quad (\text{by } p' \cdot x < w' \text{ and } \alpha < 1) \\ &= p'' \cdot x = w'', \end{aligned}$$

which contradicts $p'' \cdot x'' \leq w''$.

- Therefore, we have $p \cdot x'' < w$ or $p' \cdot x'' < w'$.
- If $p \cdot x'' < w$, then combined with $p'' \cdot x = w''$, WARP* implies that $x = x''$, and therefore, $p \cdot x < w$,

  which contradicts Walras' law.
- If $p' \cdot x'' < w'$, then combined with $p'' \cdot x' = w''$, WARP* implies that $x' = x''$, and therefore, $p' \cdot x' < w'$,

  which contradicts Walras' law.

Compensated Law of Demand

Proposition 2.5

Suppose that the demand function $x(p, w)$ satisfies Walras' law. Then $x(p, w)$ satisfies the WARP if and only if the following condition (the "compensated law of demand") holds:
For any (p, w) and (p', w') where $w' = p' \cdot x(p, w)$, we have

$$(p' - p) \cdot (x(p', w') - x(p, w)) \leq 0, \qquad (*)$$

with strict inequality whenever $x(p, w) \neq x(p', w')$.

- In particular, if $p'_\ell > p_\ell$ and $p'_k = p_k$ for all $k \neq \ell$, then $x_\ell(p', w') \leq x_\ell(p, w)$.

- The compensated law of demand is equivalently written as

  if $p' \cdot x(p,w) = w'$, then $p \cdot x(p',w') > p \cdot x(p,w)$ whenever $x(p,w) \neq x(p',w')$.

- Under Walras' law, this is equivalent to WARP by Proposition 2.4.

Slutsky Matrix

- Take any $v \in \mathbb{R}^L$ and $t > 0$.

  Let $p' = p + tv$ and $w' = p' \cdot x(p, w)$ $(= (p + tv) \cdot x(p, w))$.
- Then by $(*) \div t^2$, we have

$$\frac{1}{t} v \cdot (x(p + tv, (p + tv) \cdot x(p, w)) - x(p, w)) \leq 0.$$

- Let $t \to 0$. Then we have

$$\frac{\partial}{\partial t} v \cdot (x(p + tv, (p + tv) \cdot x(p, w)) - x(p, w))\Big|_{t=0} \leq 0.$$

- The left hand side is equal to

$$v \cdot S(p, w)v,$$

where

$$S(p, w) = D_p x(p, w) + D_w x(p, w) x(p, w)^{\mathrm{T}} \in \mathbb{R}^{L \times L}.$$

$\cdots$ Slutsky matrix

  - (ℓ, k)th entry:

$$s_{\ell k}(p, w) = \frac{\partial x_\ell}{\partial p_k}(p, w) + \frac{\partial x_\ell}{\partial w}(p, w) x_k(p, w)$$

- Thus, we have

$$v \cdot S(p, w)v \leq 0 \text{ for all } v \in \mathbb{R}^L,$$

or $S(p, w)$ is negative semi-definite.

Proposition 2.6

If the demand function $x(p, w)$ satisfies Walras' law and WARP, then for any (p, w), the Slutsky matrix $S(p, w)$ is negative semi-definite (i.e., $v \cdot S(p, w)v \leq 0$ for all $v \in \mathbb{R}^L$).

Proposition 2.7

If the demand function $x(p, w)$ is homogeneous of degree zero and satisfies Walras' law, then for any (p, w), $S(p, w)p = 0$ and $p^{\mathrm{T}} S(p, w) = 0^{\mathrm{T}}$.

- Recall from Propositions 2.1 (homogeneity), 2.2 and 2.3 (Walras' law):
  - $D_p x(p, w)p + D_w x(p, w)w = 0$
  - $x(p, w)^{\mathrm{T}} + p^{\mathrm{T}} D_p x(p, w) = 0^{\mathrm{T}}$
  - $p^{\mathrm{T}} D_w x(p, w) = 1$

- If $L = 2$, this proposition says that for all (p, w),

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

  which implies that $s_{12} = s_{21}$.

  I.e., $S(p, w)$ is symmetric if $L = 2$.

- For $L > 2$, $S(p, w)$ is not necessarily symmetric.

Exercise 2.E.1

$L = 3$

$$x_1(p,w) = \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1}$$
$$x_1(p,w) = \frac{p_3}{p_1+p_2+p_3}\frac{w}{p_2}$$
$$x_1(p,w) = \frac{p_1}{p_1+p_2+p_3}\frac{w}{p_3}$$

- Verify that $x(p,w)$ is homogeneous of degree zero and satisfies Walras' law.
- Compute s_{12} and s_{21} (where $s_{\ell k} = \frac{\partial x_\ell}{\partial p_k} + \frac{\partial x_\ell}{\partial w} x_k$).

Choice-Based versus Preference-Based

- Choice-based approach:
  1. Under homogeneity and Walras' law, WARP is equivalent to the compensated law of demand.
  2. It implies negative semi-definiteness of $S(p, w)$.
  3. These assumptions do not imply symmetry of $S(p, w)$, except in the case of $L = 2$.
- Preference-based approach:

  We will show that symmetry of $S(p, w)$ is derived by preference maximization.

Example 2.F.1

$$
\begin{aligned}
&w = 8 \\
&p^1 = (2,1,2), \quad x^1 = (1,2,2) \\
&p^2 = (2,2,1), \quad x^2 = (2,1,2) \\
&p^3 = (1,2,2), \quad x^3 = (2,2,1)
\end{aligned}
$$

- Consistent with WARP
  $(x^i \in B_{p^j,w}, x^j \neq x^i \Rightarrow x^j \notin B_{p^i,w},\ i \neq j)$
- Revealed preference relation $\succsim^*$ violates transitivity:
  $x^2 \succ^* x^1$, $x^3 \succ^* x^2$, $x^1 \succ^* x^3$