3. Classical Demand Theory

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Consumption

- Commodities $\ell = 1, \ldots, L$
- Consumption set $X \subset \mathbb{R}^L$

  In most cases, we assume $X = \mathbb{R}^L_+$.
- $x = \begin{pmatrix} x_1 \\ \vdots \\ x_L \end{pmatrix} \in X$: consumption vector/bundle
- For $x, y \in X$:
  - $x \geq y \iff x_\ell \geq y_\ell$ for all $\ell = 1, \ldots, L$
  - $x \gg y \iff x_\ell > y_\ell$ for all $\ell = 1, \ldots, L$
  - $\|x\| = \sqrt{(x_1)^2 + \cdots + (x_L)^2}$

Preference Relations

- $\succsim$: preference relation on X
  - $x \succ y \iff x \succsim y$ and $y \not\succsim x$
  - $x \sim y \iff x \succsim y$ and $y \succsim x$
- We assume that $\succsim$ is complete and transitive.
  - Completeness: for all $x, y \in X$, $x \succsim y$ or $y \succsim x$
  - Transitivity: for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Monotonicity

Definition 3.1
Let $X \subset \mathbb{R}^L$ be such that if $x \in X$ and $y \geq x$, then $y \in X$.

- $\succsim$ on X is weakly monotone if $x \succsim y$ whenever $x \geq y$.
- $\succsim$ on X is monotone if $x \succ y$ whenever $x \gg y$.
- $\succsim$ on X is strictly monotone if $x \succ y$ whenever $x \geq y$ and $x \neq y$.

Definition 3.2
$\succsim$ on X is locally nonsatiated if for any $x \in X$ and any $\varepsilon > 0$, there exists $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.

- $\succsim$ strictly monotone $\Rightarrow$ $\succsim$ monotone $\Rightarrow$ $\succsim$ locally nonsatiated

Convexity

Definition 3.3

Let $X \subset \mathbb{R}^L$ be a convex set.

- $\succsim$ on X is convex if $\alpha y + (1-\alpha)z \succsim x$ for all $\alpha \in [0,1]$ whenever $y \succsim x$ and $z \succsim x$.
- $\succsim$ on X is strictly convex if $\alpha y + (1-\alpha)z \succ x$ for all $\alpha \in (0,1)$ whenever $y \succsim x$, $z \succsim x$, and $y \neq z$.

Utility Representation

- Let $X \subset \mathbb{R}^L$ be such that if $x \in X$ and $y \geq x$, then $y \in X$.

Proposition 3.1

Suppose that $\succsim$ on X is represented by a utility function u. Then $\succsim$ is weakly monotone (strictly monotone) if and only if u is nondecreasing (strictly increasing).

- Let $X \subset \mathbb{R}^L$ be a convex set.

Proposition 3.2

Suppose that $\succsim$ on X is represented by a utility function u. Then $\succsim$ is convex (strictly convex) if and only if u is quasi-concave (strictly quasi-concave).

Existence of a Utility Function

- If X is a finite set, then a complete and transitive preference relation on X has a utility representation.
- But for infinite X, completeness and transitivity do not guarantee utility representation in general.
- Lexicographic preference relation: $X = \mathbb{R}^2_+$

  $x \succ y \iff x_1 > y_1$ or $[x_1 = y_1$ and $x_2 > y_2]$

  - If there was a utility function u that represents this $\succsim$, then the intervals $(u(x_1, 0), u(x_1, 100))$, $x_1 \in \mathbb{R}_+$, are nonempty and disjoint,

    and hence we could assign different rational numbers $q \in (u(x_1, 0), u(x_1, 100))$ for different real numbers $x_1 \in \mathbb{R}_+$, which is impossible mathematically.

Definition 3.4

$\succsim$ on X is continuous if for any sequences $\{x^m\}$ and $\{y^m\}$ in X such that $x^m \succsim y^m$ for all m, $\lim_{m\to\infty} x^m = x \in X$, and $\lim_{m\to\infty} y^m = y \in X$, we have $x \succsim y$.

Proposition 3.3

$\succsim$ on X is continuous if and only if for all $x \in X$, $\{y \in X \mid y \succsim x\}$ and $\{y \in X \mid x \succsim y\}$ are closed (relative to X).

- The lexicographic preference relation is not continuous.

Proposition 3.4

If $\succsim$ on X is continuous, then there exists a continuous utility function $u\colon X \to \mathbb{R}$ that represents $\succsim$.

Proof under Monotonicity

- Assume that $X = \mathbb{R}^L_+$, and $\succsim$ is complete, transitive, continuous, and monotone.
- Then it is weakly monotone, i.e., if $x \geq y$, then $x \succsim y$.
- Take any $x \in X$.

  By weak monotonicity, $x \succsim 0$.
- For sufficiently large $\bar{\alpha} > 0$, we have $\alpha\mathbf{1} \gg x$, hence $\alpha\mathbf{1} \succsim x$ by monotonicity (where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^L$).
- Let $A^+ = \{\alpha \in \mathbb{R}_+ \mid \alpha\mathbf{1} \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ \mid x \succsim \alpha\mathbf{1}\}$.
  - $\bar{\alpha} \in A^+$ and $\bar{\alpha} \in A^-$.
  - By completeness, $\mathbb{R}_+ = A^+ \cup A^-$.
  - By continuity, A^+ and A^- are closed.
  - Hence, by the connectedness of $\mathbb{R}_+$, $A^+ \cap A^- \neq \emptyset$.
  - By transitivity and monotonicity, $A^+ \cap A^-$ is a singleton set.

- Let $u(x)$ denote the unique element of $A^+ \cap A^-$.
- The function $x \mapsto u(x)$ represents $\succsim$:
  - Suppose that $u(x) \geq u(y)$.

    Then by weak monotonicity, $u(x)\mathbf{1} \succsim u(y)\mathbf{1}$, where $u(x)\mathbf{1} \sim x$ and $u(y)\mathbf{1} \sim y$.

    Therefore, by transitivity, $x \succsim y$.
  - Suppose that $x \succsim y$.

    Then by transitivity $u(x)\mathbf{1} \succsim u(y)\mathbf{1}$.

    Therefore, by monotonicity, $u(x) \geq u(y)$.

- The function $x \mapsto u(x)$ is continuous:
  - Take any sequence $\{x^m\}$ in X, and suppose that $x^m \to \bar{x} \in X$.

    $\{x^m\}$ is bounded: there are α^0 and α^1 such that $\alpha^1 \mathbf{1} \geq x^m \geq \alpha^0 \mathbf{1}$ for all m.
  - Take any subsequence of $\{u(x^m)\}$, denoted again by $\{u(x^m)\}$.

    $\{u(x^m)\}$ is bounded: $\alpha^1 \geq u(x^m) \geq \alpha^0$ for all m.
  - Therefore, some subsequence $\{u(x^{m(k)})\}$ converges to some α'.
  - By continuity, we have $\bar{x} \sim \alpha' \mathbf{1}$.

    But by uniqueness, we must have $\alpha' = u(\bar{x})$.
  - Thus, we have shown that any subsequence of $\{u(x^m)\}$ has a subsequence that converges to $u(\bar{x})$.
  - This implies that $\{u(x^m)\}$ itself converges to $u(\bar{x})$.

Homotheticity

Definition 3.5
Monotone $\succsim$ on $X = \mathbb{R}^L_+$ is homothetic if $\alpha x \sim \alpha y$ for all $\alpha > 0$ whenever $x \sim y$.

- A function $f\colon \mathbb{R}^L_+ \to \mathbb{R}$ is homogeneous of degree k if $f(tx) = t^k f(x)$ for all $x \in \mathbb{R}^L_+$ and $t > 0$.
- f is homogeneous if it is homogeneous of degree 1.

Proposition 3.5

1. *If monotone $\succsim$ on $\mathbb{R}^L_+$ is represented by a homogeneous utility function, then it is homothetic.*
2. *Monotone, homothetic, and continuous $\succsim$ on $\mathbb{R}^L_+$ is represented by some homogeneous utility function.*

- Verify that the utility function constructed in the proof of Proposition 3.4 under monotonicity is homogeneous when $\succsim$ is homothetic.

Quasi-Linearity

Definition 3.6

$\succsim$ on $X = \mathbb{R} \times \mathbb{R}_+^{L-1}$ is quasi-linear with respect to commodity 1 if

1. $x + \alpha e_1 \sim y + \alpha e_1$ for all $\alpha \in \mathbb{R}$ whenever $x \sim y$, and
2. $x + \alpha e_1 \succ x$ for all $x \in X$ and $\alpha > 0$.

- $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^L$

- A function $f\colon \mathbb{R}^L \to \mathbb{R}$ is quasi-linear with respect to the first coordinate if it is written as $f(x) = x_1 + \phi(x_{-1})$ for some function ϕ

  (where $x_{-1} = (x_2, \dots, x_L) \in \mathbb{R}^{L-1}$).

Proposition 3.6

1. *If $\succsim$ on $\mathbb{R} \times \mathbb{R}_+^{L-1}$ is represented by a quasi-linear utility function, then it is quasi-linear.*
2. *Quasi-linear and continuous $\succsim$ on $\mathbb{R} \times \mathbb{R}_+^{L-1}$ is represented by some quasi-linear utility function.*

Proof

- By quasi-linearity and continuity, for each $z \in \mathbb{R}_+^{L-1}$, there exists a unique $\alpha \in \mathbb{R}$ such that $(0, z) \sim \alpha e_1$.

  (Requires some topological argument.)

  Let $\phi(z) = \alpha$.

- Define the function $u\colon \mathbb{R} \times \mathbb{R}_+^{L-1} \to \mathbb{R}$ by $u(x) = x_1 + \phi(x_{-1})$.

- This function u represents $\succsim$:

$$
\begin{aligned}
& x \succsim y \\
& \iff (x_1 + \phi(x_{-1}))e_1 \succsim (y_1 + \phi(y_{-1}))e_1 \qquad \text{(by transitivity)} \\
& \iff x_1 + \phi(x_{-1}) \geq y_1 + \phi(y_{-1}) \qquad \text{(by quasi-linearity (ii))} \\
& \iff u(x) \geq u(y).
\end{aligned}
$$

Utility Maximization Problem

- In the following, we assume $X = \mathbb{R}^L_+$.
- Suppose that $\succsim$ on X is represented by a utility function u.
- For $p \gg 0$ and $w > 0$, consider the utility maximization problem:

$$\begin{aligned} \max_{x \in \mathbb{R}^L_+} \quad & u(x) \\ \text{s.t.} \quad & p \cdot x \leq w. \end{aligned} \tag{UMP}$$

- Solution correspondence $x(p, w)$ $\cdots$ Walrasian demand correspondence
- Optimal value function $v(p, w)$ $\cdots$ indirect utility function

Proposition 3.7

Suppose that u is continuous.
Then for any $p \gg 0$ and $w > 0$, (UMP) *has a solution.*

- Since $p \gg 0$ and $w > 0$, the budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ \mid p \cdot x \leq w\}$ is nonempty and compact.
- By the continuity of u, there is a maximizer by the Extreme Value Theorem.

Properties of Walrasian Demand Correspondences

Proposition 3.8

1. *Homogeneity of degree zero:*

   $x(\alpha p, \alpha w) = x(p, w)$ *for all* (p, w) *and* $\alpha > 0$.

2. *Walras' law:*

   If $\succsim$ *is locally nonsatiated, then* $p \cdot x = w$ *for all* $x \in x(p, w)$.

3. *Convexity/uniqueness:*

   If $\succsim$ *is convex, then* $x(p, w)$ *is a convex set.*

   If $\succsim$ *is strictly convex, then* $x(p, w)$ *consists of at most one element.*

Proof of Walras' law

- Suppose that $p \cdot x < w$.
- Then by local nonsatiation, there exists some $x' \in X$ close enough to x that $p \cdot x' < w$ and $x' \succ x$.
- Thus such x cannot be in $x(p, w)$.

Properties of Indirect Utility Functions

Proposition 3.9

1. $v(\alpha p, \alpha w) = v(p, w)$.
2. $v(p, w)$ *is nonincreasing in* p_ℓ *for all* ℓ *and nondecreasing in* w.

   If $\succsim$ *is locally nonsatiated, then* $v(p, w)$ *is strictly increasing in* w.
3. $v(p, w)$ *is quasi-convex in* (p, w).
4. *If* $\succsim$ *is continuous, then* $v(p, w)$ *is continuous in* (p, w).

Proof of Quasi-Convexity

- We want to show that $\{(p, w) \mid v(p, w) \leq t\}$ is a convex set for all t.
- We have

$$v(p, w) \leq t \iff [p \cdot x \leq w \Rightarrow u(x) \leq t]$$
$$\iff [u(x) > t \Rightarrow p \cdot x > w].$$

- Thus,

$$\{(p, w) \mid v(p, w) \leq t\} = \bigcap_{x: u(x) > t} \{(p, w) \mid p \cdot x > w\},$$

which is convex, since it is the intersection of convex sets $\{(p, w) \mid p \cdot x > w\}$.