7. General Equilibrium

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Framework

- L commodities
- Consumers: $1, \dots, I$

  Each consumer $i = 1, \dots, I$ is characterized by:
  - consumption set $X_i \subset \mathbb{R}^L$ (usually $X_i = \mathbb{R}^L_+$)
  - preference relation $\succsim_i$ on X_i
  - We assume that $\succsim_i$ is complete and transitive for all i.
- Firms: $1, \dots, J$

  Each firm $j = 1, \dots, J$ is characterized by:
  - production set $Y_j \subset \mathbb{R}^L$
  - We assume that Y_j is nonempty and closed for all j.
- Initial endowments: $\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_L) \in \mathbb{R}^L$

Feasible Allocations

- Allocation:
  $(x, y) = ((x_1, \ldots, x_I), (y_1, \ldots, y_J)) \in \prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j$
  - $x_i \in X_i$: consumer i's consumption vector
  - $y_j \in Y_j$: firm j's production vector
  - $(\prod_{i=1}^{I} X_i = X_1 \times \cdots \times X_I$, $\prod_{j=1}^{J} Y_j = Y_1 \times \cdots \times Y_J)$

Definition 7.1

An allocation (x, y) is feasible if $\sum_i x_i = \bar{\omega} + \sum_j y_j$.

- Denote the set of all feasible allocations by A.

Pareto Efficiency

Definition 7.2

1. For $x, x' \in \prod_{i=1}^{I} X_i$, x' Pareto dominates x if

$$x'_i \succsim_i x_i \text{ for all } i = 1, \ldots, I,$$
$$x'_i \succ_i x_i \text{ for some } i = 1, \ldots, I.$$

2. A feasible allocation $(x, y) \in A$ is Pareto efficient if there exists no feasible allocation $(x', y') \in A$ such that x' Pareto dominates x.

Private Ownership Economies

- A private ownership economy:
  $\mathcal{E} = ((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, (\omega_i, \theta_{i1}, \dots, \theta_{iJ})_{i=1}^I)$ where:
  - $(X_i, \succsim_i)$: consumer i's preference relation
  - Y_j: firm j's production set
  - $\omega_i \in X_i$: consumer i's initial endowment, where $\bar{\omega} = \sum_i \omega_i$
  - $\theta_{ij} \in [0, 1]$: share of consumer i's claim to the profit of firm j, where $\sum_i \theta_{ij} = 1$ for all j

Definition 7.3

A Walrasian equilibrium of a private ownership economy $\mathcal{E}$ is $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)) \in \mathbb{R}^L \times \prod_i X_i \times \prod_j Y_j$ such that

1. [Profit maximization]
   for every $j = 1, \ldots, J$, y_j^* maximizes the profit $p^* \cdot y_j$ in Y_j,

   i.e., $y_j^* \in Y_j \quad$ and $\quad p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_j$;

2. [Preference maximality]
   for every $i = 1, \ldots, I$, x_i^* is maximal for $\succsim_i$ in the budget set

   $$B_i = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot \omega_i + \textstyle\sum_j \theta_{ij}(p^* \cdot y_j^*)\},$$

   i.e., $x_i^* \in B_i \quad$ and $\quad x_i^* \succsim_i x_i$ for all $x_i \in B_i$;

3. [Market clearing]
   $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$.

Pure Exchange Economies

- A private ownership economy $\mathcal{E} = ((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, (\omega_i, \theta_i)_{i=1}^I)$ is called a pure exchange economy if $X_i = \mathbb{R}_+^L$ for all i, and $J = 1$ and $Y_1 = -\mathbb{R}_+^L$.
  - $((x_i)_{i=1}^I, y_1)$ is feasible for some $y_j \in Y_j$ if and only if $\sum_i x_i - \sum_i \omega_i \leq 0$.
  - If $y_j(p) \neq \emptyset$, then it must be that $p \geq 0$ and $\pi_j(p) = 0$.
- We denote a pure exchange economy by $\mathcal{E}' = ((\succsim_i)_{i=1}^I, (\omega_i)_{i=1}^I)$.
- We define Walrasian equilibrium of a pure exchange economy $\mathcal{E}' = ((\succsim_i)_{i=1}^I, (\omega_i)_{i=1}^I)$ as follows. $\rightarrow$

Definition 7.4

A Walrasian equilibrium of a pure exchange economy $\mathcal{E}'$ is $(p^*, (x_i^*)_{i=1}^I,) \in \mathbb{R}^L \times (\mathbb{R}_+^L)^I$ such that

1. $p^* \geq 0$;
2. for every $i = 1, \ldots, I$, x_i^* is maximal for $\succsim_i$ in the budget set $B_i = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot \omega_i\}$,

   i.e., $x_i^* \in B_i$ and $x_i^* \succsim_i x_i$ for all $x_i \in B_i$;
3. $\sum_i x_i^* \leq \sum_i \omega_i$ and $p^* \cdot (\sum_i x_i^* - \sum_i \omega_i) = 0$.

- Given $p^* \geq 0$, an equivalent expression of condition 3 is:

  $\sum_i x_i^* \leq \sum_i \omega_i$, and $p_\ell^* = 0$ if $\sum_i x_{i\ell}^* < \sum_i \omega_{i\ell}$.

Proposition 7.1

$(p^*, (x_i^*)_{i=1}^I)$ is a Walrasian equilibrium of $\mathcal{E}'$ if and only if $(p^*, (x_i^*)_{i=1}^I, y_1^*)$ is a Walrasian equilibrium of $\mathcal{E}$ for some y_1^*.

Proof of the "only if" part

- Suppose that $(p^*, (x_i^*)_{i=1}^I)$ is a Walrasian equilibrium of $\mathcal{E}'$.
- Let $y_1^* = \sum_{i=1}^I x_i^* - \sum_{i=1}^I \omega_i$ (≤ 0).
- Then $y_1^* \in Y_1$ and $p^* \cdot y_1^* = 0$, so $y_1^* \in y_1(p^*)$.

Example: Edgeworth Box

Example: One-Consumer, One-Producer Economy

- $L = 2$
  - $\ell = 1$: leisure (price w)
  - $\ell = 2$: consumption good (price p)
- $J = 1$: production function $y = f(z)$
  - $\ell = 1$: input (z)
  - $\ell = 2$: output (y)
- $I = 1$: utility function $u(x_1, x_2)$
  Endowment: $\omega_1 = (\bar{L}, 0)$

Exercise 15.C.2

- $f(z) = z^{\frac{1}{2}}$
- $u(x_1, x_2) = \log x_1 + \log x_2$
- $\bar{L} = 1$

First Fundamental Theorem of Welfare Economics

- "A Walrasian equilibrium allocation is Pareto efficient."
- The assumption of local nonsatiation is necessary.

Proposition 7.2

In a private ownership economy $\mathcal{E} = ((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, (\omega_i, \theta_i)_{i=1}^I)$*, assume that for each* i*,* $\succsim_i$ *is locally nonsatiated.*
If $(p^*, ((x_i^*)_{i=1}^I, (y_j^* i)_{j=1}^J))$ *is a Walrasian equilibrium of* $\mathcal{E}$*, then* $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ *is Pareto efficient.*

Lemma 7.3

Assume that $\succsim_i$ is locally nonsatiated.
If x_i^* is maximal for $\succsim_i$ in $B(p, w_i)$, then $p \cdot x_i \geq w_i$ whenever $x_i \succsim_i x_i^*$.

Proof

- If $p \cdot x_i < w_i$, then by local nonsatiation, there exists some $\tilde{x}_i$ close to x_i such that $p \cdot \tilde{x}_i < w_i$ and $\tilde{x}_i \succ_i x_i$.
- By preference maximality, $x_i^* \succsim_i \tilde{x}_i$, and hence $x_i^* \succ_i x_i$.

Proof of Proposition 7.2

Suppose that $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J))$ is a Walrasian equilibrium of $\mathcal{E}$.

Step 1

- Write $w_i^* = p^* \cdot \omega_i + \sum_{j=1}^J \theta_{ij}(p^* \cdot y_j^*)$.
- Then

$$\begin{aligned}\sum_i w_i^* &= \sum_i p^* \cdot \omega_i + \sum_j \underbrace{\sum_i \theta_{ij}}_{=1}(p^* \cdot y_j^*) \\ &= \sum_i p^* \cdot \omega_i + \sum_j p^* \cdot y_j^*.\end{aligned}$$

Step 2

If an allocation $((x_i)_{i=1}^{I}, (y_j)_{j=1}^{J})$ Parato dominates $((x_i^*)_{i=1}^{I}, (y_j^*)_{j=1}^{J})$ and $(y_j)_{j=1}^{J}$ is feasible (i.e., $y_j \in Y_j$ for all j), then

$$\sum_{i=1}^{I} p^* \cdot x_i > \sum_{i=1}^{I} p^* \cdot \omega_i + \sum_{j=1}^{J} p^* \cdot y_j.$$

- By definition,
  - (i) $x_i \succsim_i x_i^*$ for all i, and
  - (ii) $x_i \succ_i x_i^*$ for some i.

- By maximality of x_i^* in $B(p^*, w_i^*)$,

$$x_i \succ_i x_i^* \Rightarrow p^* \cdot x_i > w_i^*.$$

- By maximality of x_i^* in $B(p^*, w_i^*)$ and local nonsatiation of $\succsim_i$,

$$x_i \succsim_i x_i^* \Rightarrow p^* \cdot x_i \geq w_i^*$$

  (by Lemma 7.3).

- Therefore, by (i) and (ii),

  (i′) $p^* \cdot x_i \geq w_i^*$ for all i, and

  (ii′) $p^* \cdot x_i > w_i^*$ for some i.

- Hence, we have

$$\sum_i p^* \cdot x_i > \sum_i w_i^* = \sum_i p^* \cdot \omega_i + \sum_j p^* \cdot y_j^*.$$

- By optimality of y_j^* and $y_j \in Y_j$,
  we have $p^* \cdot y_j^* \geq p^* \cdot y_j$ for all j.
- Therefore, we have

$$\sum_i p^* \cdot x_i > \sum_i p^* \cdot \omega_i + \sum_j p^* \cdot y_j.$$

Step 3

- But for any feasible allocation $((x_i)_{i=1}^I, (y_j)_{j=1}^J)$, we must have

$$\sum_i p^* \cdot x_i = \sum_i p^* \cdot \omega_i + \sum_j p^* \cdot y_j.$$

- Hence, Step 2 implies that

  if allocation $((x_i)_{i=1}^I, (y_j)_{j=1}^J)$ Parato dominates $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$, then it is not feasible.

- Thus, we have shown that $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ is Pareto efficient.

Equilibrium Concepts

Definition 7.5

A price equilibrium with transfers of $((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, \bar{\omega})$ is $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)) \in \mathbb{R}^L \times \prod_i X_i \times \prod_j Y_j$ such that there exists $(w_1, \ldots, w_I)$ with $\sum_i w_i = p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^*$ such that

1. [Profit maximization]
   for every $j = 1, \ldots, J$, y_j^* maximizes the profit $p^* \cdot y_j$ in Y_j;
2. [Preference maximality]
   for every $i = 1, \ldots, I$, x_i^* is maximal for $\succsim_i$ in the budget set
   $$\{x_i \in X_i \mid p^* \cdot x_i \leq w_i\},$$
   or equivalently, $p^* \cdot x_i^* \leq w_i$, and if $x_i \succ_i x_i^*$, then $p^* \cdot x_i > w_i$;
3. [Market clearing]
   $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

- If $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J))$ is a Walrasian equilibrium of $((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, (\omega_i, \theta_i)_{i=1}^I)$, then it is a price equilibrium with transfers of $((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, \bar{\omega})$ (where $\bar{\omega} = \sum_i \omega_i$).
  - Let $w_i = p^* \cdot \omega_i + \sum_{j=1}^J \theta_{ij}(p^* \cdot y_j^*)$.
- The proof of Proposition 7.2 in fact proves that (under local nonsatiation) the allocation of a price equilibrium with transfers is Pareto efficient.

Definition 7.6

A price quasi-equilibrium with transfers of $((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, \bar{\omega})$ is $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)) \in \mathbb{R}^L \times \prod_i X_i \times \prod_j Y_j$ such that there exists $(w_1, \ldots, w_I)$ with $\sum_i w_i = p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^*$ such that

1. [Profit maximization]
   for every $j = 1, \ldots, J$, y_j^* maximizes the profit $p^* \cdot y_j$ in Y_j;
2. for every $i = 1, \ldots, I$, $p^* \cdot x_i^* \leq w_i$, and if $x_i \succ_i x_i^*$, then $p^* \cdot x_i \geq w_i$;
3. [Market clearing]
   $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

- If $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J))$ is a price equilibrium with transfers, then it is a price quasi-equilibrium with transfers.

Second Fundamental Theorem of Welfare Economics

- Under convexity assumptions,

  "any Pareto efficient allocation is supported as a price quasi-equilibrium with transfers".

Proposition 7.4

In an economy $\mathcal{E} = ((X_i, \succsim_i)_{i=1}^I, (Y_j)_{j=1}^J, \bar{\omega})$, assume that

- *for every $j = 1, \ldots, J$, Y_j is convex; and*
- *for every $i = 1, \ldots, I$, X_i is convex and $\succsim_i$ is convex and locally nonsatiated.*

Then for any Pareto efficient feasible allocation $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$, there exists $p^* \neq 0$ such that $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J))$ is a price quasi-equilibrium with transfers of $\mathcal{E}$.

Proof

- Suppose that feasible allocation $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$ is Pareto efficient.

Step 1

- For each i, define

$$V_i = \{x_i \in X_i \mid x_i \succ_i x_i^*\}.$$

- V_i is a convex set:
  - Take any $x_i, x_i' \in V_i$ and $\alpha \in [0,1]$, where $x_i \succ_i x_i^*$ and $x_i' \succ_i x_i^*$.
  - By completeness, $x_i \succsim_i x_i'$ or $x_i' \succsim_i x_i$.

    Assume the former without loss of generality.
  - By convexity of $\succsim_i$, we have $\alpha x_i + (1-\alpha)x_i' \succsim_i x_i'$.
  - By transitivity, we have $\alpha x_i + (1-\alpha)x_i' \succ_i x_i^*$; thus $\alpha x_i + (1-\alpha)x_i' \in V_i$.

Step 2

- Define

$$V = \sum_i V_i = \{\sum_i x_i \in \mathbb{R}^L \mid x_1 \in V_1, \ldots, x_I \in V_I\},$$

which is a convex set (it is the sum of convex sets).

- Define

$$Y = \sum_j Y_j = \{\sum_j y_j \in \mathbb{R}^L \mid y_1 \in Y_1, \ldots, y_J \in Y_J\},$$

which is a convex set by convexity of $Y_1, \ldots, Y_J$.

Step 3

- $V \cap (\{\bar{\omega}\} + Y) = \emptyset$:
  - Suppose $V \cap (\{\bar{\omega}\} + Y) \neq \emptyset$, and let $z \in V \cap (\{\bar{\omega}\} + Y)$.
  - Then we have $z = \sum_i x_i$ for some $x_1 \in V_1, \dots, x_I \in V_I$ and $z = \bar{\omega} + \sum_j y_j$ for some $y_1 \in Y_1, \dots, y_J \in Y_J$,

    which means that there exists a feasible allocation $((x_i)_{i=1}^I, (y_j)_{j=1}^J)$ that Pareto dominates $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$.
  - This contradicts Pareto efficiency of $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$.

Step 4

- Since X and $\{\bar{\omega}\} + Y$ are convex sets and $V \cap (\{\bar{\omega}\} + Y) = \emptyset$, by the Separating Hyperplane Theorem (Proposition 6.6), there exist $p^* \neq 0$ and c such that

$$p^* \cdot z \leq c \leq p^* \cdot z' \text{ for all } z \in \{\bar{\omega}\} + Y \text{ and } z' \in V. \qquad (*)$$

Step 5

- If $x_i \succsim_i x_i^*$ for all i, then $p^* \cdot \sum_i x_i \geq c$:
  - Suppose that $x_i \succsim_i x_i^*$ for all i.
  - By local nonsatiation, for each i there exists $\hat{x}_i \in X_i$ arbitrarily close to x_i such that $\hat{x}_i \succ_i x_i$.
  - By transitivity, $\hat{x}_i \succ_i x_i^*$, i.e., $\hat{x}_i \in V_i$.
  - Thus, $\sum_i \hat{x}_i \in V$, and $p^* \cdot \sum_i \hat{x}_i \geq c$ by $(*)$.
  - Letting $\hat{x}_i \to x_i$, we have $p^* \cdot \sum_i x_i \geq c$.

Step 6

- $p^* \cdot \sum_i x_i^* = p^* \cdot (\bar{\omega} + \sum_j y_j^*) = c$:
  - By Step 5, $p^* \cdot \sum_i x_i^* \geq c$.
  - By $(*)$, $p^* \cdot (\bar{\omega} + \sum_j y_j^*) \leq c$.
  - By feasibility, $p^* \cdot \sum_i x_i^* = p^* \cdot (\bar{\omega} + \sum_j y_j^*)$.

Step 7

- For every j, $p^* \cdot y_j \leq p^* \cdot y_j^*$ for all $y_j \in Y_j$:
  - Fix any j and take any $y_j \in Y_j$.
  - Since $y_j + \sum_{h \neq j} y_h^* \in Y$, by $(*)$ and Step 6 we have

$$p^* \cdot (\bar{\omega} + y_j + \textstyle\sum_{h \neq j} y_h^*) \leq c = p^* \cdot (\bar{\omega} + y_j^* + \textstyle\sum_{h \neq j} y_h^*),$$

    and hence $p^* \cdot y_j \leq p^* \cdot y_j^*$.

Step 8

- For every i, if $x_i \succ_i x_i^*$, then $p^* \cdot x_i \geq p^* \cdot x_i^*$:
  - Fix any i and suppose that $x_i \succ_i x_i^*$.
  - By Steps 5 and 6, we have

$$p^* \cdot (x_i + \textstyle\sum_{k \neq i} x_k^*) \geq c = p^* \cdot (x_i^* + \textstyle\sum_{k \neq i} x_k^*),$$

    and hence $p^* \cdot x_i \geq p^* \cdot x_i^*$.

Step 9

- With $w_i = p^* \cdot x_i^*$ for all i, $(p^*, ((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J))$ is a price quasi-equilibrium with transfers:
  - Condition 1 follows from Step 7.
  - Condition 2 follows from Step 8.
  - Condition 3 follows from feasibility of $((x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J)$.

Equilibrium versus Quasi-Equilibrium

- A price equilibrium with transfers is a price quasi-equilibrium with transfers, but the converse does not hold in general.
- The converse holds, for example if for all i, $p^* \cdot x_i^* > 0$ and $0 \in X_i$.
- More generally:

Proposition 7.5

Assume that X_i is convex and $\succsim_i$ is continuous.
Let $x_i^* \in X_i$, p, and w_i be such that $x_i \succ_i x_i^* \Rightarrow p \cdot x_i \geq w_i$.
Then if there exists $x_i' \in X_i$ such that $p \cdot x_i' < w_i$, then $x_i \succ_i x_i^* \Rightarrow p \cdot x_i > w_i$.

Proof

- Assume that for some $x_i \in X_i$, we have $x_i \succ_i x_i^*$ and $p \cdot x_i \leq w_i$.
- Then by continuity of $\succsim_i$, for $\alpha < 1$ sufficiently close to 1 we have $\alpha x_i + (1-\alpha)x_i' \succ_i x_i^*$

  (where $\alpha x_i + (1-\alpha)x_i' \in X_i$ by convexity of X_i).
- But then we have

$$p \cdot (\alpha x_i + (1-\alpha)x_i') = \alpha(p \cdot x_i) + (1-\alpha)(p \cdot x_i') < w_i,$$

  which contradicts "$x_i \succ_i x_i^* \Rightarrow p \cdot x_i \geq w_i$".

Economies with Quasi-Linear Preferences

- Commodities: $1, \dots, L$
  consumption $x_i \in \mathbb{R}^L$, production $y_j \in \mathbb{R}^L$
- Commodity 0 (numeraire for all consumers)
  consumption $m_i \in \mathbb{R}$, input $z_j \in \mathbb{R}$
- Preferences: for each i, $\succsim_i$ is represented by
  $u_i(m_i, x_i) = m_i + \phi_i(x_i) \quad (m_i \in \mathbb{R},\ x_i \in X_i \subset \mathbb{R}^L)$
  - Locally nonsatiated $\Rightarrow$ Walras' law
  - Strictly increasing in m_i $\Rightarrow$ Any Walrasian equilibrium price of commodity 0 must be strictly positive.
  - We will normalize prices so that $p_0 = 1$.
- Endowments: $(\omega_{i0}, \omega_i) \in \mathbb{R} \times X_i$
- Production: for each j, $Y_j \subset \mathbb{R}^{1+L}$
  production vector $(-z_j, y_j) \in Y_j$

Equilibrium

Proposition 7.6

$((1, p^*), ((m_i^*, x_i^*)_{i=1}^I, (-z_j^*, y_j^*)_{j=1}^J) \in \mathbb{R}^{1+L} \times \prod_i (\mathbb{R} \times X_i) \times \prod_j Y_j$ *is a price equilibrium with transfers if and only if there exists* $(w_1, \ldots, w_I)$ *with* $\sum_i w_i = (\bar{\omega}_0 + p^* \cdot \bar{\omega}) + \sum_j (-z_j^* + p^* \cdot y_j^*)$ *such that*

1. *for every* j, $(-z_j^*, y_j^*)$ *solves* $\max_{(-z_j, y_j) \in Y_j} -z_j + p^* \cdot y_j$;
2. *for every* i, x_i^* *solves* $\max_{x_i \in X_i} \phi_i(x_i) - p^* \cdot x_i$, *and* $m_i^* = w_i - p^* \cdot x_i^*$;
3. $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

- By Walras' law, the market clearing for commodity 0 is automatically satisfied.
- The components other than $(m_i^*)_{i=1}^I$ do not depend on the choice of $(w_1, \ldots, w_I)$.

Fundamental Theorems

- For each i, $\succsim_i$ is locally nonsatiated.
- The First Fundamental Theorem holds.
- If
  - for every j, Y_j is a convex set, and
  - for every i, X_i is a convex set and ϕ_i is a concave function,

  then the Second Fundamental Theorem holds.
- Price equilibrium and price quasi-equilibrium are equivalent.

Two-Commodity Case—Partial Equilibrium Analysis

- Two commodities
  - commodity $\ell \cdots$ price p
  - commodity 0: numeraire ("the other commodities") $\cdots$ price 1
- Production
  - c_j: firm j's cost function
    $c_j' > 0$, $c_j'' > 0$, $c_j(0) = 0$
  - $Y_j = \{(-z_j, q_j) \in \mathbb{R}^2 \mid z_j \geq c_j(q_j),\ q_j \geq 0\}$
  - Profit maximization: $\max_{q_j} pq_j - c_j(q_j)$
    $\Rightarrow p \leq c_j'(q_j^*)$ with "$=$" if $q_j^* > 0$
  - Supply function for ℓ:
    $y_i(p) = (c_j')^{-1}(p)$ if $p > c_j'(0)$
  - $z_j^* = c_j(q_j^*)$

- Consumption
  - Utility function:
    $u_i(m_i, x_i) = m_i + \phi_i(x_i) \quad (m_i \in \mathbb{R},\ x_i \in \mathbb{R}_+)$
    $\phi_j' > 0$, $\phi_j'' < 0$, $\phi_j(0) = 0$
  - $\omega_{im} > 0$, $\omega_{i\ell} = 0$
  - Utility maximization:
    $\max_{m_i, x_i} m_i + \phi_i(x_i)$
    subject to $m_i + px_i \leq \omega_{im} + \sum_j \theta_{ij}(pq_j^* - c_j(q_j^*))$
    $\Rightarrow \phi_i'(x_i^*) \leq p$ with "$=$" if $x_i^* > 0$
  - Demand function for ℓ:
    $x_i(p) = (\phi_i')^{-1}(p)$ if $p < \phi_i'(0)$
  - $m_i^* = \omega_{im} + \sum_j \theta_{ij}(pq_j^* - c_j(q_j^*)) - px_i^*$

Equilibrium

- $(p^*, ((x_i^*)_{i=1}^I, (q_j^*)_{j=1}^J)) \in \mathbb{R} \times \mathbb{R}_+^I \times \mathbb{R}_+^J$ is a price equilibrium with transfers if and only if
  1. for every j, $p^* \leq c_j'(q_j^*)$ with "$=$" if $q_j^* > 0$;
  2. for every i, $\phi_i'(x_i^*) \leq p^*$ with "$=$" if $x_i^* > 0$;
  3. $\sum_i x_i^* = \sum_j q_j^*$.

Surplus Maximization

- Consumer surplus of i:

$$
\begin{aligned}
CS_i &= \int_0^{x_i^*} \phi_i'(x_i)dx_i - p^* x_i^* \\
&= \phi_i(x_i^*) - \phi_i(0) - p^* x_i^* = \phi_i(x_i^*) - p^* x_i^*
\end{aligned}
$$

- Total surplus:

$$
\begin{aligned}
&\textstyle\sum_i (\phi_i(x_i^*) - p^* x_i^*) + \sum_j (p^* q_j^* - c_j(q_j^*)) \\
&= \textstyle\sum_i \phi_i(x_i^*) - \sum_j c_j(q_j^*) \qquad \text{(by market clearing)}
\end{aligned}
$$

- Total surplus maximization:

$$
\begin{aligned}
\max \quad & \textstyle\sum_i \phi_i(x_i) - \sum_j c_j(q_j) \\
\text{s.t.} \quad & \textstyle\sum_i x_i - \sum_j q_j = 0 \\
& x_i \geq 0, \;\; q_j \geq 0
\end{aligned}
$$

- Lagrangian:

  $L = \sum_i \phi_i(x_i) - \sum_j c_j(q_j) + \mu(\sum_j q_j - \sum_i x_i)$

- KKT condition:

  There exists μ such that

  1. for every j, $\mu \leq c_j'(q_j)$ with "=" if $q_j > 0$;
  2. for every i, $\phi_i'(x_i) \leq \mu$ with "=" if $x_i > 0$;
  3. $\sum_i x_i = \sum_j q_j$.

- Hence:

  $(p^*, ((x_i^*)_{i=1}^I, (q_j^*)_{j=1}^J))$ is a price equilibrium for some p^*
  if and only if $((x_i^*)_{i=1}^I, (q_j^*)_{j=1}^J))$ is total surplus maximizing.

Pareto Efficiency

- Consider the maximization problem:

$$\begin{aligned} \max \quad & m_1 + \phi_1(x_1) \\ \text{s.t.} \quad & m_i + \phi_i(x_i) \geq \bar{u}_i \quad (i = 2, \ldots, I) \\ & \textstyle\sum_i x_i - \sum_j q_j \leq 0 \\ & \textstyle\sum_i m_i + \sum_j z_j \leq \bar{\omega}_m \\ & z_j \geq c_j(q_j) \quad (j = 1, \ldots, J) \\ & x_i \geq 0, \ q_j \geq 0 \end{aligned}$$

- Lagrangian:

$$\begin{aligned} L = {} & m_1 + \phi_1(x_1) + \textstyle\sum_{i \neq 1} \lambda_i (m_i + \phi_i(x_i) - \bar{u}_i) \\ & + \mu(\textstyle\sum_j q_j - \sum_i x_i) + \eta(\bar{\omega}_m - \sum_i m_i - \sum_j z_j) \\ & + \textstyle\sum_j \nu_j (z_j - c_j(q_j)) \end{aligned}$$

- KKT condition:
  - $1 = \eta$
  - $\lambda_i = \eta$ for all $i \neq 1$
  - $\phi_1'(x_1) \leq \mu$ with "=" if $x_1 > 0$
  - $\lambda_i \phi_i'(x_i) \leq \mu$ with "=" if $x_i > 0$ for all $i \neq 1$
  - $\mu \leq \nu_j c_j'(q_j)$ with "=" if $q_j > 0$ for all j
  - $\eta = \nu_j$ for all j
- which is equivalent to:
  - $1 = \eta = \lambda_2 = \cdots \lambda_I = \nu_1 = \cdots = \nu_J$
  - $\phi_i'(x_i) \leq \mu$ with "=" if $x_i > 0$ for all i
  - $\mu \leq c_j'(q_j)$ with "=" if $q_j > 0$ for all j

- Hence:

  $(p^*, ((x_i^*)_{i=1}^I, (q_j^*)_{j=1}^J))$ is a price equilibrium for some p^* if and only if $((m_i^*, x_i^*)_{i=1}^I, (z_j^*, q_j^*)_{j=1}^J))$ is Pareto efficient for some $(m_i^*)_{i=1}^I$ and $(z_j^*)_{j=1}^J))$.

Existence of Walrasian Equilibrium

- We only consider a simple case of a pure exchange economy $\mathcal{E} = ((\succsim_i)_{i=1}^I, (\omega_i)_{i=1}^I)$:
  - For each i, $\succsim_i$ is a complete and transitive preference relation on $X_i = \mathbb{R}_+^L$.
  - Assume that $\sum_i \omega_i \gg 0$.
- $(p^*, (x_i^*)_{i=1}^I,) \in \mathbb{R}^L \times (\mathbb{R}_+^L)^I$ is a Walrasian equilibrium of $\mathcal{E}$ if
  - $p^* \geq 0$;
  - for every $i = 1, \ldots, I$, x_i^* is maximal for $\succsim_i$ in the budget set $B_i(p^*, p^* \cdot \omega_i)$;
  - $\sum_i x_i^* \leq \sum_i \omega_i$ and $p^* \cdot (\sum_i x_i^* - \sum_i \omega_i) = 0$.

Assumptions

In the following, we assume:

(a) For each i, $\succsim_i$ is continuous and strictly convex.

$\Rightarrow$ Demand *function* $x_i(\cdot)$ is well defined and continuous for $p \gg 0$.

(b) For each i, $\succsim_i$ is locally nonsatiated.

$\Rightarrow$ Walras' law holds: $p \cdot (x_i(p, p \cdot \omega_i) - \omega_i) = 0$ for any $p \gg 0$.

Excess Demand Functions

- Excess demand function of i:

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i \qquad (p \gg 0)$$

- (Aggregate) excess demand function:

$$z(p) = \sum_i z_i(p) = \sum_i x_i(p, p \cdot \omega_i) - \sum_i \omega_i \qquad (p \gg 0)$$

- Properties:
  1. $z(\cdot)$ is continuous.
  2. $z(\cdot)$ is homogeneous of degree zero.
  3. $p \cdot z(p) = 0$ for all $p \gg 0$ (Walras' law).

Proposition 7.7

Assume (a) and (b).
$p^* \gg 0$ *is a Walrasian equilibrium price vector if and only if* $z(p^*) \leq 0$.

Proof of the "if" part

- Suppose that $z(p^*) \leq 0$.
- Let $x_i^* = x_i^*(p^*, p^* \cdot \omega_i)$ for each i.
- $\sum_i x_i^* \leq \sum_i \omega_i$ holds by assumption,
  while $p^* \cdot (\sum_i x_i^* - \sum_i \omega_i) = 0$ holds by Walras' law.

Equilibrium Existence: Version 1

We strengthen (b) to:

(c) For each i, $\succsim_i$ is strongly monotone.

$\Rightarrow$ p^* is a Walrasian equilibrium price vector if and only if $p^* \gg 0$ and $z(p^*) = 0$.

Proposition 7.8

Assume (a) and (c).
Then a Walrasian equilibrium of $\mathcal{E}$ exists.

- Proof: See the proof of Proposition 17.C.1 in MWG,

  which uses "Kakutani's fixed point theorem".

Equilibrium Existence: Version 2

We drop (c) and assume:

(d) For each i, $z_i(p)$ is well defined for all $p \in \mathbb{R}^L_+ \setminus \{0\}$ and is continuous on $\mathbb{R}^L_+ \setminus \{0\}$.

$\Rightarrow$ Walras' law holds for all $p \in \mathbb{R}^L_+ \setminus \{0\}$.

$p^* \in \mathbb{R}^L_+ \setminus \{0\}$ is a Walrasian equilibrium price vector if and only if $z(p^*) \leq 0$.

Proposition 7.9

Assume (a), (b), and (d).
Then a Walrasian equilibrium of $\mathcal{E}$ exists.

- For proof, we will use "Brouwer's fixed point theorem".

Brouwer's Fixed Point Theorem

Proposition 7.10 (Brouwer's Fixed Point Theorem)

Suppose that $X \subset \mathbb{R}^N$ *is a nonempty, compact, and convex set, and that* $f\colon X \to X$ *is a continuous function from* X *into itself. Then* f *has a fixed point, i.e., there exists* $x \in X$ *such that* $x = f(x)$*.*

- What if X is not compact?
- What if X is not convex?
- What if f is not continuous?

Proof of Proposition 7.9

- We want to show that there exists $p^* \in \mathbb{R}^L_+ \setminus \{0\}$ such that $z(p^*) \leq 0$.
- Let $\Delta = \{p \in \mathbb{R}^L_+ \mid p_1 + \cdots + p_L = 1\}$,

  which is nonempty, compact, and convex.
- It suffices to show that there exists $p^* \in \Delta$ such that $z(p^*) \leq 0$.
- Define the function $z^+(p) = (z_1^+(p), \ldots, z_L^+(p))$ by $z_\ell^+(p) = \max\{z_\ell(p), 0\}$.
- $z^+(p)$ is a continuous function.
- Define the function $f\colon \Delta \to \Delta$ by

$$f_\ell(p) = \frac{p_\ell + z_\ell^+(p)}{\sum_{k=1}^L (p_k + z_k^+(p))} \qquad (\ell = 1, \ldots, L).$$

- f is a continuous function from the nonempty, compact, and convex set Δ to Δ.
- Thus, by Brouwer's fixed point theorem, f has a fixed point $p^* \in \Delta$:

$$p_\ell^* = \frac{p_\ell^* + z_\ell^+(p^*)}{\sum_{k=1}^{L}(p_k^* + z_k^+(p^*))} \qquad (\ell = 1, \ldots, L).$$

- We show that p^* satisfies $z(p^*) \leq 0$.

- By Walras' law, we have

$$0 = \sum_\ell p^*_\ell z_\ell(p^*) = \frac{\sum_\ell (p^*_\ell z_\ell(p^*) + z^+_\ell(p^*) z_\ell(p^*))}{\sum_{k=1}^L (p^*_k + z^+_k(p^*))}$$
$$= \frac{\sum_\ell z^+_\ell(p^*) z_\ell(p^*)}{\sum_{k=1}^L (p^*_k + z^+_k(p^*))},$$

  and therefore $\sum_\ell z^+_\ell(p^*) z_\ell(p^*) = 0$.

- Since

$$z^+_\ell(p^*) z_\ell(p^*) = \begin{cases} z_\ell(p^*)^2 > 0 & \text{if } z_\ell(p^*) > 0, \\ 0 & \text{if } z_\ell(p^*) \leq 0, \end{cases}$$

  it follows from $\sum_\ell z^+_\ell(p^*) z_\ell(p^*) = 0$ that $z_\ell(p^*) \leq 0$ for all $\ell = 1, \ldots, L$, as desired.