8. General Equilibrium under Uncertainty

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Contingent Commodities

- $\ell = 1, \ldots, L$: physical commodities
- $i = 1, \ldots, I$: consumers
- $j = 1, \ldots, J$: firms
- $s = 1, \ldots, S$: states of the world
- State-contingent commodity (ℓ, s):

  a title to receive a unit of commodity ℓ when state s is realized.
- State-contingent commodity vector:
  $x = (x_{11}, \ldots, x_{L1}, \ldots, x_{1S}, \ldots, x_{LS}) \in \mathbb{R}^{LS}$

- Endowments for consumer i:
  $\omega_i = (\omega_{11}, \ldots, \omega_{L1}, \ldots, \omega_{1S}, \ldots, \omega_{LS}) \in \mathbb{R}^{LS}$
- $\succsim_i$: consumer i's preference relation on a consumption set $X_i \subset \mathbb{R}^{LS}$
- $Y_j \subset \mathbb{R}^{LS}$: firm j's production set
- $y_j \in Y_j$: state-contingent production plan
- θ_{ij}: share of firm j owned by consumer i
  (state independent, for simplicity)

Assumption

- For every contingent commodity (ℓ, s), there is a market with price $p_{\ell s}$.
- These markets open before uncertainty is resolved.

Arrow-Debreu Equilibrium

Definition 8.1
$(p^*, (x_i^*)_{i=1}^I, (y_j^*)_{j=1}^J) \in \mathbb{R}^{LS} \times \prod_{i=1}^I X_i \times \prod_{j=1}^J Y_j$
is an *Arrow-Debreu equilibrium* if

1. for each j, $p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_j$;
2. for each i, $x_i^* \succsim_i x_i$ for all $x_i \in \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p^* \cdot y_j^*\}$; and
3. $\sum_{i=1}^I x_i^* = \sum_{j=1}^J y_j^* + \sum_{i=1}^I \omega_i$.

- This is just a particular case of Walrasian equilibrium.

  The Welfare Theorems hold under the usual assumptions.

Example 1

- $I = 2$ $(i = A, B)$, $L = 1$, $S = \{1, 2\}$
- $\omega_A = (\omega_{1A}, \omega_{2A}) = (1, 0)$
  $\omega_B = (\omega_{1B}, \omega_{2B}) = (0, 1)$
  - $\bar{\omega}_s = \omega_{sA} + \omega_{sB} = 1$ for all $s \in S$

    $\cdots$ There is no aggregate uncertainty
- $\succsim_i$ is represented by

$$\pi_{1i} u_i(x_{1i}) + \pi_{2i} u_i(x_{2i})$$

  - π_{si}: i's subjective probability of state $s \in S$
  - $u_i' > 0$, $u_i'' < 0$
  - $MRS_{12i}(x_{1i}, x_{2i}) = \frac{\pi_{1i} u_i'(x_{1i})}{\pi_{2i} u_i'(x_{2i})}$

- $\max \pi_{1i} u_i(x_{1i}) + \pi_{2i} u_i(x_{2i})$

  subject to $p_1 x_{1i} + p_2 x_{1i} \leq p_1 \omega_{1i} + p_2 \omega_{2i}$

- Equilibrium conditions:
  - $\frac{\pi_{1A}}{\pi_{2A}} \frac{u'_A(x_{1A})}{u'_A(x_{2A})} = \frac{p_1}{p_2} = \frac{\pi_{1B}}{\pi_{2B}} \frac{u'_B(x_{1B})}{u'_B(x_{2B})}$
  - $x_{1A} + x_{1B} = \bar{\omega}_1$ $(= 1)$

    $x_{2A} + x_{2B} = \bar{\omega}_2$ $(= 1)$

Case (a): $\dfrac{\pi_{1A}}{\pi_{2A}} = \dfrac{\pi_{1B}}{\pi_{2B}}$

- On the contract curve:
  $$\frac{u'_A(x_{1A})}{u'_A(x_{2A})} = \frac{u'_B(\bar{\omega}_1 - x_{1A})}{u'_B(\bar{\omega}_2 - x_{2A})}, \quad \bar{\omega}_1 = \bar{\omega}_2 = 1$$
  $$\Rightarrow x_{1A} = x_{2A}$$

- In the equilibrium:
  $$\frac{p_1^*}{p_2^*} = \frac{\pi_{1A}}{\pi_{2A}} = \frac{\pi_{1B}}{\pi_{2B}}$$
  $x_{1i}^* = x_{2i}^*$ $\cdots$ consumers insure completely

Case (b): $\frac{\pi_{1A}}{\pi_{2A}} < \frac{\pi_{1B}}{\pi_{2B}}$

- On 45 degree line:

  $$MRS_{12i} = \frac{\pi_{1i}}{\pi_{2i}}$$

- In the equilibrium:

  $$\frac{\pi_{1A}}{\pi_{2A}} < \frac{p_1^*}{p_2^*} < \frac{\pi_{1B}}{\pi_{2B}}$$

  $x_{1A}^* < x_{2A}^*$, $x_{1B}^* > x_{2B}^*$

  $\cdots$ consumer's consumption is higher in the state he thinks more likely (relative to the other's beliefs)

Example 2

- Same as in Example 1 Case (a) except:

  $$\omega_A = (\omega_{1A}, \omega_{2A}) = (2, 0)$$
  $$\omega_B = (\omega_{1B}, \omega_{2B}) = (0, 1)$$

  - $\bar{\omega}_1 = 2 > \bar{\omega}_2 = 1$

    $\cdots$ There is aggregate uncertainty

- $\dfrac{\pi_{1A}}{\pi_{2A}} = \dfrac{\pi_{1B}}{\pi_{2B}} = \dfrac{\pi_1}{\pi_2}$

- On the contract curve:
  $MRS_{12i} < \frac{\pi_1}{\pi_2}$ for each $i = A, B$
  - On the contract curve:
    $\frac{\pi_1}{\pi_2}\frac{u'_A(x_{1A})}{u'_A(x_{2A})} = \frac{\pi_1}{\pi_2}\frac{u'_B(\bar{\omega}_1 - x_{1A})}{u'_B(\bar{\omega}_2 - x_{2A})}$
  - If $x_{1A} \leq x_{2A}$:
    - $\frac{u'_A(x_{1A})}{u'_A(x_{2A})} \geq 1$ (by concavity)
    - $\bar{\omega}_1 - x_{1A} > (\bar{\omega}_2 - x_{2A}$ (since $\bar{\omega}_1 > \bar{\omega}_2$)
      $\Rightarrow \frac{u'_B(\bar{\omega}_1 - x_{1A})}{u'_B(\bar{\omega}_2 - x_{2A})} < 1$ (by concavity)
  - Therefore, $x_{1A} > x_{2A}$, and hence $\frac{u'_A(x_{1A})}{u'_A(x_{2A})} < 1$ (by concavity)
- In the equilibrium: $\frac{p_1^*}{p_2^*} < \frac{\pi_1}{\pi_2}$
  - In particular, if $\pi_1 = \pi_2$, then $p_1^* < p_2^*$

Asset Markets

- $\ell = 1, \ldots, L$: physical commodities
- $i = 1, \ldots, I$: consumers
- $s = 1, \ldots, S$: states of the world
- $\succsim_i$: i's preference relation on $\mathbb{R}_+^{LS}$
  with a utility function representation U_i
  (assumed to be strongly monotone)

- After uncertainty is resolved, spot markets open at $t = 1$.
- A price vector at state s is denoted by $p_s \in \mathbb{R}^L$,
  and the overall price vector by $p \in \mathbb{R}^{LS}$.

Assets

Asset markets open at $t = 0$.

We consider *real* assets,
where returns are in units of commodity 1.

- An *asset* is identified with its return vector:

$$r = (r_1, \ldots, r_S)' \in \mathbb{R}^S.$$

  (Here we always consider vectors as column vectors.)

- Examples:
  - $\mathbf{1} = (1, \ldots, 1)'$: "commodity futures"
  - $e_s = (0, \ldots, 0, 1, 0, \ldots, 0)'$ (sth unit vector): called an "Arrow security".

Example: Derivative Assets

- The *call option* on an asset $r \in \mathbb{R}^S$ ("primary asset") at the strike price $c \in \mathbb{R}$:

$$r(c) = (\max\{0, r_1 - c\}, \dots, \max\{0, r_S - c\})'.$$

  It gives the option to buy a unit of r at price c *after the state is realized*.

- For example, if $S = 4$ and $r = (4, 3, 2, 1)'$,

$$r(3.5) = (0.5, 0, 0, 0)',$$
$$r(2.5) = (1.5, 0.5, 0, 0)',$$
$$r(1.5) = (2.5, 1.5, 0.5, 0)'.$$

Return Matrix

- We fix K assets, $r_1, \dots, r_K \in \mathbb{R}^S$, as given.

  We assume that $r_k \geq 0$, $r_k \neq 0$ for all k.

- The $S \times K$ matrix

$$R = \begin{pmatrix} r_1 & \cdots & r_K \end{pmatrix} = \begin{pmatrix} r_{11} & \cdots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{S1} & \cdots & r_{SK} \end{pmatrix}$$

  is called the *return matrix*.

- A vector of trades in these assets, $z = (z_1, \dots, z_K)' \in \mathbb{R}^K$, is called a *portfolio*.

- An asset price vector is denoted by $q = (q_1, \dots, q_K)' \in \mathbb{R}^K$.

Equilibrium

Definition 8.2
$(q, p, (z_i^*)_{i=1}^I, (x_i^*)_{i=1}^I) \in \mathbb{R}^K \times \mathbb{R}^{LS} \times (\mathbb{R}^K)^I \times (\mathbb{R}_+^{LS})^I$
is a *Radner equilibrium* if:

(i) for all i, (z_i^*, x_i^*) solves

$$\max_{z_i \in \mathbb{R}^K,\, x_i \in \mathbb{R}_+^{LS}} U_i(x_i)$$

$$\text{s.t.} \quad \sum_k q_k z_{ki} \leq 0$$

$$p_s' x_{si} \leq p_s' \omega_{si} + \sum_k p_{1s} z_{ki} r_{sk} \text{ for all } s;$$

(ii) $\sum_i z_i^* \leq 0$ and $\sum_i x_i^* \leq \sum_i \omega_i$.

Price Normalization and Budget Constraint

- Normalize $p_{1s} = 1$ for all s.
- Budget constraint of i:

$$B_i(q, p, R) = \{x_i \in \mathbb{R}_+^{LS} \mid \exists\, z_i \in R^K \text{ s.t. } q' z_i \leq 0 \text{ and } m_i \leq R z_i\},$$

where

$$m_i = (p_1'(x_{1i} - \omega_{1i}), \ldots, p_S'(x_{Si} - \omega_{Si}))' \in \mathbb{R}^S.$$

State Prices

Proposition 8.1

If $q \in \mathbb{R}^K$ is an asset price vector in a Radner equilibrium, then there exists $\mu \in \mathbb{R}^S_{++}$ such that $q' = \mu' R$.

- μ is called a *state price vector*.
- μ_s is the shadow price of the state-contingent commodity for state s.
- $q' = \mu' R \iff$

$$
\begin{aligned}
\begin{pmatrix} q_1 & \cdots & q_K \end{pmatrix} &= \begin{pmatrix} \mu_1 & \cdots & \mu_S \end{pmatrix} \begin{pmatrix} r_{11} & \cdots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{S1} & \cdots & r_{SK} \end{pmatrix} \\
&= \begin{pmatrix} \sum_s \mu_s r_{s1} & \cdots & \sum_s \mu_s r_{sK} \end{pmatrix}.
\end{aligned}
$$

Proof 1 (1/2)

- $q \in \mathbb{R}^K$ is *arbitrage free* if there is no portfolio $z \in \mathbb{R}^K$ such that $q'z \leq 0$, $Rz \geq 0$, and $[q'z \neq 0$ or $Rz \neq 0]$.
- Under our assumption that $r_k \geq 0$, $r_k \neq 0$ for all k, an arbitrage free price vector must be strictly positive, and hence the above definition is equivalent to the definition in MWG:

  $q \in \mathbb{R}^K$ is arbitrage free if and only if there is no portfolio $z \in \mathbb{R}^K$ such that $q'z \leq 0$, $Rz \geq 0$, and $Rz \neq 0$.

  (I.e., there is no portfolio that is budgetarily feasible and that yields a nonnegative return in every state and a strictly positive return in some state.)
- Under strongly monotone preferences, an equilibrium asset price vector $q \in \mathbb{R}^K$ is arbitrage free.
- Proposition 8.1 follows from the following lemma.

Proof 1 (2/2)

Lemma 8.2

For any $R \in \mathbb{R}^{S \times K}$*,*
$q \in \mathbb{R}^K$ *is arbitrage free if and only if*
there exists $\mu \in \mathbb{R}^S_{++}$ *such that* $q' = \mu' R$*.*

- Proof by "Stiemke's Lemma".

Proof 2 (1/2)

- Choose any consumer i. Assume that U_i has a representation $U_i(x_{1i}, \dots, x_{Si}) = \sum_s \pi_{si} u_{si}(x_{si}) \qquad (\pi_{si} > 0)$ where u_{si} are concave, strictly increasing, and differentiable.
- Denote by v_{si} the indirect utility function derived from u_{si}.
- Let q, p be the equilibrium prices, and consider

$$\max_{z_i \in \mathbb{R}^K} \quad \sum_s \pi_{si} v_{si}(p_s, p_s' \omega_{si} + \sum_k r_{sk} z_{ki})$$
$$\text{s.t.} \quad \sum_k q_k z_{ki} \leq 0.$$

- The equilibrium portfolio plan z_i^* must satisfy the FOC with some $\alpha_i > 0$ (Lagrange multiplier):

$$\sum_s \pi_{si} \frac{\partial v_{si}}{\partial w_{si}}(p_s, w_{si}^*)\, r_{sk} = \alpha_i q_k \text{ for all } k,$$

where $w_{si}^* = p_s' \omega_{si} + \sum_k r_{sk} z_{ki}^*$.

Proof 2 (2/2)

- Define $\mu \in \mathbb{R}^S_{++}$ by

$$\mu_s = \frac{\pi_{si}}{\alpha_i} \frac{\partial v_{si}}{\partial w_{si}}(p_s, w^*_{si}).$$

- This satisfies $q' = \mu' R$.
- Note: choice of a different consumer may lead to a different μ.

Complete Markets

Definition 8.3

An asset structure with an $S \times K$ return matrix R is *complete* if $\operatorname{rank} R = S$, i.e.,

$$\{v \in \mathbb{R}^S \mid v = Rz \text{ for some } z \in \mathbb{R}^K\} = \mathbb{R}^S.$$

- Example:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(where all the Arrow securities are available) is complete.

- Example:

$$R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

  is not complete.

  No portfolio can give, for example, a return vector $(0, 0, 1)'$.

Equivalence between Radner and Arrow-Debreu Equilibria

Proposition 8.3

Assume that the asset structure is complete.

(i) *If* $(p, x^*) \in \mathbb{R}^{LS}_{++} \times (\mathbb{R}^{LS}_{+})^I$ *is an Arrow-Debreu equilibrium, then there* $q \in \mathbb{R}^K_{++}$ *and* $z^* \in (\mathbb{R}^K)^I$ *such that* (q, p, z^*, x^*) *is a Radner equilibrium.*

(ii) *If* $(q, p, z^*, x^*) \in \mathbb{R}^K_{++} \times \mathbb{R}^{LS}_{++} \times (\mathbb{R}^K)^I \times (\mathbb{R}^{LS}_{+})^I$ *is a Radner equilibrium, then there exists* $\mu \in \mathbb{R}^S_{++}$ *such that* $((\mu_1 p_1, \ldots, \mu_S p_S), x^*)$ *is an Arrow-Debreu equilibrium.*

Sketch of the Proof (1/4)

- Denote

$$B_i^{\mathrm{AD}}(p) = \{x_i \in \mathbb{R}_+^{LS} \mid \textstyle\sum_s p_s'(x_{si} - \omega_{si}) \leq 0\}$$

and

$$B_i^{\mathrm{R}}(q, p) = \{x_i \in \mathbb{R}_+^{LS} \mid \exists\, z_i \in R^K \text{ s.t. } q' z_i \leq 0 \text{ and } m_i \leq \Lambda R z_i\},$$

where

$$m_i = (p_1'(x_{1i} - \omega_{1i}), \dots, p_S'(x_{Si} - \omega_{Si}))' \in \mathbb{R}^S.$$

Sketch of the Proof (2/4)

(i) Let (p, x^*) be an Arrow-Debreu equilibrium.

- Denote

$$\Lambda = \begin{pmatrix} p_{11} & & 0 \\ & \ddots & \\ 0 & & p_{1S} \end{pmatrix}.$$

Then

$$\Lambda R = \begin{pmatrix} p_{11}r_{11} & \cdots & p_{11}r_{1K} \\ \vdots & \ddots & \vdots \\ p_{1S}r_{S1} & \cdots & p_{1S}r_{SK} \end{pmatrix}.$$

Let

$$q' = \mathbf{1}'\Lambda R \qquad\qquad (\iff q_k = \textstyle\sum_s p_{1s}r_{sk} \;\forall\, k).$$

Sketch of the Proof (3/4)

- WTS: $x_i^* \in B_i^{\mathrm{R}}(q,p)$ and $x_i \in B_i^{\mathrm{R}}(q,p) \Rightarrow x_i \in B_i^{\mathrm{AD}}(p)$.
- Let

$$m_i^* = (p_1'(x_{1i}^* - \omega_{1i}), \ldots, p_S'(x_{Si}^* - \omega_{Si}))' \in \mathbb{R}^S.$$

- Since $\operatorname{rank} \Lambda R = S$ by completeness,
  for each $i = 1, \ldots, I-1$, there exists z_i^* such that

$$m_i^* = \Lambda R z_i^*.$$

  Define

$$z_I^* = -(z_1^* + \cdots + z_{I-1}^*).$$

- Show $x_i^* \in B_i^{\mathrm{R}}(q,p)$.

Sketch of the Proof (4/4)

(ii) Let (q, p, z^*, x^*) be a Radner equilibrium.

Assume without loss of generality that $p_{1s} = 1$ for all s.

- By Proposition 8.1, there exists $\mu \in \mathbb{R}^S_{++}$ such that $q' = \mu' R$.
- WTS: $x_i^* \in B_i^{\mathrm{AD}}(\mu_1 p_1, \ldots, \mu_S p_S)$ and
  $x_i \in B_i^{\mathrm{AD}}(\mu_1 p_1, \ldots, \mu_S p_S) \Rightarrow x_i \in B_i^{\mathrm{R}}(q, p)$.
- For the former,

$$\sum_s \mu_s p_s'(x_{si} - \omega_{si}) \leq \sum_s \mu_s (R z_i)_s = \mu' R z_i = q' z_i \leq 0.$$

- For the latter,
  by the completeness, there exists z_i such that $m_i = R z_i$.

  Then,

$$q' z_i = \mu' R z_i = \mu' m_i \leq 0.$$