0. Basic Mathemetics

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Microeconomics I

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Notations

- $\mathbb{N} = \{1, 2, 3, \ldots\}$: the set of natural numbers
- $\mathbb{R}$: the set of real numbers
- $\mathbb{R}_+$: the set of nonnegative real numbers
- $\mathbb{R}_{++}$: the set of positive real numbers
- $\mathbb{R}^N$: the set of N-dimensional vectors
- $\mathbb{R}^N_+$: the set of N-dimensional nonnegative vectors
- $\mathbb{R}^N_{++}$: the set of N-dimensional positive vectors

- For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$,

$$\|x\| = \sqrt{(x_1)^2 + \cdots + (x_N)^2}.$$

- For $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$,

$$p \cdot x = p_1 x_1 + \cdots + p_N x_N.$$

(Sometimes written as "px" without "$\cdot$")

Maximum/Minimum and Supremum/Infimum

- For $A \subset \mathbb{R}$, $A \neq \emptyset$,
  - $x \in \mathbb{R}$ is the maximum of A, denoted $\max A$, if
    - $x \in A$, and
    - $y \leq x$ for all $y \in A$;
  - $x \in \mathbb{R}$ is the minimum of A, denoted $\min A$, if
    - $x \in A$, and
    - $x \leq y$ for all $y \in A$.

- Supremum
  = "generalization" of maximum (least upper bound)
- Infimum
  = "generalization" of minimum (greatest lower bound)
- Example:
  - For $A = [0, 1]$: $\sup A = 1$
    (Note $\sup A \in A$, so $\sup A = \max A$)
  - For $A = [0, 1)$: $\sup A = 1$
    (A has no maximum.)
  - For $A = [0, \infty)$: A has no supremum.
    (Sometimes we write $\sup A = \infty$.)

- Any nonempty subset A of $\mathbb{R}$ that is bounded above has a (finite) supremum.
- Any nonempty subset A of $\mathbb{R}$ that is bounded below has a (finite) infimum.
- In general $\sup A \notin A$, but $\sup A$ can be "approached arbitrarily closely" by elements of A.

Convergence in $\mathbb{R}^N$

- A *sequence* in $\mathbb{R}^N$ is a function from $\mathbb{N}$ to $\mathbb{R}^N$.

  A sequence is denoted by $\{x^m\}_{m=1}^{\infty}$, or simply $\{x^m\}$, or x^m.

Definition 0.1

A sequence $\{x^m\}_{m=1}^{\infty}$ *converges* to $\bar{x} \in \mathbb{R}^N$ if
for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\|x^m - \bar{x}\| < \varepsilon \text{ for all } m \geq M.$$

In this case, we write $\lim_{m\to\infty} x^m = \bar{x}$ or $x^m \to \bar{x}$ (as $m \to \infty$).

- $\bar{x}$ is called the *limit* of $\{x^m\}_{m=1}^{\infty}$.
- A sequence that converges to some $\bar{x} \in \mathbb{R}^N$ is said to be *convergent*.

Examples/Properties

- The sequence $\{\frac{1}{m}\}_{m=1}^{\infty}$ in $\mathbb{R}$ converges to 0.
- For sequences $\{x^m\}_{m=1}^{\infty}$ and $\{y^m\}_{m=1}^{\infty}$ in $\mathbb{R}^N$,

  if $x^m \to x$ and $y^m \to y$, then $x^m + y^m \to x + y$ and $cx^m \to cx$ for any $c \in \mathbb{R}$.
- For sequences $\{x^m\}_{m=1}^{\infty}$, $\{y^m\}_{m=1}^{\infty}$, and $\{z^m\}_{m=1}^{\infty}$ in $\mathbb{R}$,
  - if $x^m \leq y^m$ for all m and if $x^m \to x$ and $y^m \to y$, then $x \leq y$;
  - if $x^m \leq y^m \leq z^m$ for all m and if $x^m \to x$ and $z^m \to x$, then $y^m \to x$.

Open Sets and Closed Sets in $\mathbb{R}^N$

- For $x \in \mathbb{R}^N$, the ε-open ball around x:

$$B_\varepsilon(x) = \{y \in \mathbb{R}^N \mid \|y - x\| < \varepsilon\}.$$

Definition 0.2

- $A \subset \mathbb{R}^N$ is an open set if for any $x \in A$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$.
- $A \subset \mathbb{R}^N$ is a closed set if $\mathbb{R}^N \setminus A$ is an open set.

Examples

- In $\mathbb{R}$,
  - $(0,1)$: open
  - $[0,1]$: closed
  - $(0,1]$: not open, not closed
- In $\mathbb{R}^N$, $\emptyset$ and $\mathbb{R}^N$ are open and closed.
- In $\mathbb{R}^N$, $\mathbb{R}^N_+$ is closed.
- For $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$ and $w \in \mathbb{R}$,

  $$\{x \in \mathbb{R}^N_+ \mid p \cdot x \leq w\}$$

  is a closed set.

Relative Openness and Relative Closedness

Definition 0.3

For $X \subset \mathbb{R}^N$,

- $A \subset X$ is an open set relative to X if for any $x \in A$, there exists $\varepsilon > 0$ such that $(B_\varepsilon(x) \cap X) \subset A$.
- $A \subset X$ is a closed set relative to X if $X \setminus A$ is an open set relative to X.

Characterization of Closed Sets

Proposition 0.1

The following statements are equivalent:

1. $A \subset \mathbb{R}^N$ *is a closed set.*
2. *For any sequence* $\{x^m\}_{m=1}^{\infty}$ *in* A *with* $x^m \to \bar{x} \in \mathbb{R}^N$, *we have* $\bar{x} \in A$.

Proof

$1 \Rightarrow 2$

- We prove its contrapositive "not $2 \Rightarrow$ not 1".
- Suppose that there exist a sequence $\{x^m\}$ in A and $\bar{x} \in \mathbb{R}^N$ such that $x^m \to \bar{x}$ and $\bar{x} \notin A$ (i.e., $\bar{x} \in \mathbb{R}^N \setminus A$).
- Since $x^m \to \bar{x}$, for any $\varepsilon > 0$ we have $x^m \in B_\varepsilon(\bar{x})$ for some m, where $x^m \in A$.
- This means that $B_\varepsilon(\bar{x}) \cap A \neq \emptyset$, i.e., $B_\varepsilon(\bar{x}) \not\subset \mathbb{R}^N \setminus A$.
- We have shown that there exists some $\bar{x} \in \mathbb{R}^N \setminus A$ such that $B_\varepsilon(\bar{x}) \not\subset \mathbb{R}^N \setminus A$ for any $\varepsilon > 0$.
- Hence, $\mathbb{R}^N \setminus A$ is not open, i.e., A is not closed.

$2 \Rightarrow 1$

- We prove its contrapositive "not $1 \Rightarrow$ not 2".
- Suppose that A is not closed, i.e., $\mathbb{R}^N \setminus A$ is not open.
- Then there exists some $\bar{x} \in \mathbb{R}^N \setminus A$ such that for any $\varepsilon > 0$, we have $B_\varepsilon(\bar{x}) \not\subset \mathbb{R}^N \setminus A$, i.e., $B_\varepsilon(\bar{x}) \cap A \neq \emptyset$.
- For each $m \in \mathbb{N}$, take any $x^m \in B_{\frac{1}{m}}(\bar{x}) \cap A$.
- Then $\{x^m\}$ is a sequence in A and converges to $\bar{x} \notin A$.
- We have shown that there exists a convergent sequence in A whose limit is not in A.

Characterization of (Relative) Closed Sets

Proposition 0.2

For $X \subset \mathbb{R}^N$, the following statements are equivalent:

1. $A \subset X$ *is a closed set relative to* X.
2. *For any sequence* $\{x^m\}_{m=1}^{\infty}$ *in* A *with* $x^m \to \bar{x} \in X$, *we have* $\bar{x} \in A$.

Compact Sets

- $A \subset \mathbb{R}^N$ is *bounded* if there exists $r \in \mathbb{R}$ such that $\|x\| < r$ for all $x \in A$.

Definition 0.4

$A \subset \mathbb{R}^N$ is compact if it is bounded and closed.

Examples:

- $[0, 1] \subset \mathbb{R}$ is compact.
- $[0, \infty) \subset \mathbb{R}$ is not compact.
- $(0, 1] \subset \mathbb{R}$ is not compact.
- For $p \in \mathbb{R}^N_{++}$, $\{x \in \mathbb{R}^N_+ \mid p \cdot x \le w\}$ is compact.

Characterizations of Compact Sets

Proposition 0.3

For $A \subset \mathbb{R}^N$, the following statements are equivalent:

1. A *is compact.*
2. *For every sequence* $\{x^m\} \subset A$*,*
   there exist a subsequence $\{x^{m(k)}\}$ *of* $\{x^m\}$ *and a point* $x \in A$
   such that $x^{m(k)} \to x$*.*

Characterizations of Compact Sets

- A family $\{F_\lambda\}_{\lambda \in \Lambda}$ of subsets of $\mathbb{R}^N$ satisfies the finite intersection property if for any finite subfamily $\{F_{\lambda_i}\}_{i=1,\ldots,k}$ of $\{F_\lambda\}_{\lambda \in \Lambda}$, we have $\bigcap_{i=1}^k F_{\lambda_i} \neq \emptyset$.

Proposition 0.4

For $A \subset \mathbb{R}^N$, the following statements are equivalent:

1. $A \subset \mathbb{R}^N$ *is compact.*
2. *For any family $\{F_\lambda\}_{\lambda \in \Lambda}$ of closed (relative to A) subsets of A that satisfies the finite intersection property, we have $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.*

Continuous Functions

Let X be a nonempty subset of $\mathbb{R}^N$.

Definition 0.5

- A function $f\colon X \to \mathbb{R}^K$ is continuous at $\bar{x} \in X$ if for any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x}$ as $m \to \infty$, we have $f(x^m) \to f(\bar{x})$ as $m \to \infty$

  (i.e., $\lim_{m\to\infty} f(x^m) = f(\lim_{m\to\infty} x^m)$).

- $f\colon X \to \mathbb{R}^K$ is continuous if it is continuous at all $\bar{x} \in X$.

Examples/Properties

- For $p \in \mathbb{R}^N$, the function $f\colon \mathbb{R}^N \to \mathbb{R}$ defined by $f(x) = p \cdot x$ is continuous.
- Suppose that $f\colon X \to \mathbb{R}$ is a continuous function.

  For any $c \in \mathbb{R}$,

  - $\{x \in X \mid f(x) \geq c\}$ and $\{x \in X \mid f(x) \leq c\}$ are closed sets relative to X.
  - $\{x \in X \mid f(x) > c\}$ and $\{x \in X \mid f(x) < c\}$ are open sets relative to X.

- Proof of closedness of $A = \{x \in X \mid f(x) \geq c\}$:
  - Take any sequence $\{x^m\}$ in A, and assume that $x^m \to \bar{x} \in X$.
  - Then $f(x^m) \geq c$ for all m, but by the continuity of f, we have $f(x^m) \to f(\bar{x})$.
  - Therefore, we have $f(\bar{x}) \geq c$, which means that $\bar{x} \in A$.
  - Therefore, A is a closed set (relative to X).

Extreme Value Theorem

Proposition 0.5

If $X \subset \mathbb{R}^N$ *is a nonempty compact set and*
$f\colon X \to \mathbb{R}$ *is a continuous function,*
then f *has a maximizer and a minimizer, i.e.,*
there exist $x^*, x^{**} \in X$ *such that* $f(x^{**}) \leq f(x) \leq f(x^*)\ \forall\, x \in X$*.*

Proof

- For each $x \in X$, define

$$F_x = \{y \in X \mid f(y) \geq f(x)\}.$$

- By the continuity of f, F_x is a closed set (relative to X) for each $x \in X$.
- $\{F_x\}_{x \in X}$ satisfies the finite intersection property:
  - Take any finite subset $\{x^1, \ldots, x^k\}$ of X.
  - Let $i^* = 1, \ldots, k$ be such that $f(x^{i^*}) = \max\{f(x^1), \ldots, f(x^k)\}$.
  - Then $x^{i^*} \in F_{x^i}$ for all $i = 1, \ldots, k$.
  - Therefore, $\bigcap_{i=1}^k F_{x^i} \neq \emptyset$.

- Since X is a compact set, we therefore have $\bigcap_{x\in X} F_x \neq \emptyset$ by Proposition 0.4.
- Take any $x^* \in \bigcap_{x\in X} F_x$.

  It satisfies $x^* \in X$ and $f(x^*) \geq f(x)$ for all $x \in X$, which means that it is a maximizer of f on X.