6. Production

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Production Sets

- $Y \subset \mathbb{R}^L$: production set
- For $(y_1, \ldots, y_L) \in Y$
  - $y_\ell > 0 \Rightarrow \ell$: output
  - $y_\ell < 0 \Rightarrow \ell$: input
- Example: Suppose $L = 5$ and $y = (-5, 2, -6, 3, 0) \in Y$.
  - revenue $= p_2 \times y_2 + p_4 \times y_4$
  - cost $= p_1 \times (-y_1) + p_3 \times (-y_3)$
  - profit $= [p_2 \times y_2 + p_4 \times y_4] - [p_1 \times (-y_1) + p_3 \times (-y_3)] = p \cdot y$
- If a production function $f\colon \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is given where commodity L is the output, then

$$Y = \{(-z_1, \ldots, -z_{L-1}, q) \mid q \leq f(z_1, \ldots, z_{L-1}),\ z_\ell \geq 0\}.$$

Properties of Production Sets

1. Y is nonempty.
2. Y is closed.
3. No free lunch: $Y \cap \mathbb{R}^L_+ \subset \{0\}$
4. Possibility of inaction: $0 \in Y$
5. Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$, or equivalently, $Y - \mathbb{R}^L_+ \subset Y$.

   ($A - B = \{c \mid c = a - b$ for some $a \in A$ and $b \in B\}$)
6. Irreversibility: If $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

Properties of Production Sets

7. Nonincreasing returns to scale:
   If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \in [0, 1]$.

8. Nondecreasing returns to scale:
   If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \geq 1$.

9. Constant returns to scale:
   If $y \in Y$, then $\alpha y \in Y$ for all $\alpha \geq 0$.

   (I.e., Y is a *cone*.)

10. Additivity: $Y + Y \subset Y$.

11. Convexity:
    If $y, y' \in Y$, then $\alpha y + (1 - \alpha) y' \in Y$ for all $\alpha \in [0, 1]$.

12. Y is a *convex cone*:
    If $y, y' \in Y$, then $\alpha y + \beta y' \in Y$ for all $\alpha \geq 0$ and $\beta \geq 0$.

Convexity

Proposition 6.1

Y is additive and exhibits nonincreasing returns to scale if and only if it is a convex cone.

Constant Returns to Scale

Proposition 6.2

If $Y \subset \mathbb{R}^L$ is convex and $0 \in Y$, then there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L \mid (y, -1) \in Y'\}$.

- Decreasing returns reflect the scarcity of some underlying, unlisted input ("entrepreneurial factor").

Proof

- Let $Y' = \{y' \in \mathbb{R}^{L+1} \mid y' = \alpha(y, -1) \text{ for some } y \in Y \text{ and } \alpha \geq 0\}$.

Profit Maximization Problem

$$\max_{y} \; p \cdot y \qquad \text{(PMP)}$$
$$\text{s. t.} \; y \in Y$$

- Supply correspondence:

$$y(p) = \arg\max_{y \in Y} \; p \cdot y$$
$$= \{y \in \mathbb{R}^L \mid y \in Y \text{ and } p \cdot y \geq p \cdot y' \text{ for all } y' \in Y\}$$

- Profit function:

$$\pi(p) = \max_{y \in Y} \; p \cdot y$$

- Analogous to expenditure minimization!

Properties of π and y

Proposition 6.3

Suppose Y is nonempty and closed.

1. *π is homogeneous of degree one.*
2. *π is convex.*
3. *If Y is convex and satisfies free disposal, then $Y = \{y \in \mathbb{R}^L \mid p \cdot y \leq \pi(p) \text{ for all } p \geq 0\}$.*
4. *y is homogeneous of degree zero.*
5. *If Y is convex, then $y(p)$ is a convex set for all p.*
6. *[Hotelling's lemma] If $y(p)$ is a singleton, then $\nabla \pi(p) = y(p)$.*
7. *If y is a continuously differentiable function, then $Dy(p)$ is symmetric and positive semi-definite, and $Dy(p)p = 0$.*

Separating Hyperplane Theorems

Proposition 6.4 (Strong Separating Hyperplane Theorem)

Suppose that $C \subset \mathbb{R}^N$, $C \neq \emptyset$, *is convex and closed, and that* $b \notin C$.

Then there exist $p \in \mathbb{R}^N$ *with* $p \neq 0$ *and* $c \in \mathbb{R}$ *such that*

$$p \cdot y \leq c < p \cdot b \text{ for all } y \in C.$$

Proposition 6.5 (Supporting Hyperplane Theorem)

Suppose that $C \subset \mathbb{R}^N$, $C \neq \emptyset$, *is convex, and that* $b \notin C$. *Then there exists* $p \in \mathbb{R}^N$ *with* $p \neq 0$ *such that*

$p \cdot y \leq p \cdot b$ *for all* $y \in C$.

Proposition 6.6 (Separating Hyperplane Theorem)

Suppose that $A, B \subset \mathbb{R}^N$*,* $A, B \neq \emptyset$*, are convex, and that* $A \cap B = \emptyset$*.*
Then there exists $p \in \mathbb{R}^N$ *with* $p \neq 0$ *such that*

$$p \cdot x \leq p \cdot y \text{ for all } x \in A \text{ and } y \in B.$$

Proof

- Since A and B are convex,
  $A - B = \{x - y \in \mathbb{R}^N \mid x \in A,\ y \in B\}$ is also convex.
- Since $A \cap B = \emptyset$, $0 \notin A - B$.
- Thus by the Supporting Hyperplane Theorem, there exists $p \in \mathbb{R}^N$ with $p \neq 0$ such that

  $$p \cdot z \leq p \cdot 0 \text{ for all } z \in A - B,$$

  or

  $$p \cdot x \leq p \cdot y \text{ for all } x \in A \text{ and } y \in B.$$

Separation with Nonnegative/Positive Vectors

Lemma 6.7

For $A \subset \mathbb{R}^N$, $A \neq \emptyset$, suppose that $A - \mathbb{R}^N_{++} \subset A$.
For $p \in \mathbb{R}^N$, if there exists $c \in \mathbb{R}$ such that $p \cdot x \leq c$ for all $x \in A$, then $p \geq 0$.

Proposition 6.8

Suppose that $C \subset \mathbb{R}^N$, $C \neq \emptyset$, is convex.
If $C \cap \mathbb{R}^N_{++} = \emptyset$, then there exists $p \geq 0$ with $p \neq 0$ such that

$$p \cdot x \leq 0 \text{ for all } x \in C.$$

Proof

- Consider the convex set $A = C - \mathbb{R}^N_{++}$, where $0 \notin A$.

Support Function of a Convex Set

For $Y \subset \mathbb{R}^N$, $Y \neq \emptyset$, define the function $\phi_Y \colon \mathbb{R}^N \to (-\infty, \infty]$ by

$$\phi_Y(p) = \sup_{y \in Y} p \cdot y.$$

In mathematics, this is called the *support function* of Y.

Proposition 6.9

Let $Y \subset \mathbb{R}^N$, $Y \neq \emptyset$, be a closed convex set. Then

$$Y = \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \in \mathbb{R}^N\}.$$

Proof

- $Y \subset$ (RHS): By definition.
- $Y \supset$ (RHS):

  Let $\bar{y} \notin Y$.
- Since Y is closed and convex, by the Separating Hyperplane Theorem, there exist $\bar{p} \neq 0$ and $c \in \mathbb{R}$ such that

  $$\bar{p} \cdot y \leq c < \bar{p} \cdot \bar{y} \text{ for all } y \in Y,$$

  and hence

  $$\phi_Y(\bar{p}) = \sup_{y \in Y} \bar{p} \cdot y \leq c < \bar{p} \cdot \bar{y}.$$
- This means that $\bar{y} \notin$ (RHS).

From Profit Function to Production Set

- What additional assumptions are needed to recover Y from the profit function, which is defined only for nonnegative, or positive, price vectors (where we allow the profit function to take values in $(-\infty, \infty]$)?

Proposition 6.10

If Y is nonempty, convex, and closed and satisfies free disposal, then

$$Y = \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \geq 0\}.$$

- Generally, $Y \subsetneq \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \gg 0\}$.

Proof

- $Y \subset$ (RHS): Immediate.
- $Y^c \subset (\text{RHS})^c$: Suppose that $\bar{y} \notin Y$.
- Since Y is nonempty, convex, and closed, there exist $\bar{p} \neq 0$ and c such that

  $$\bar{p} \cdot y \leq c < \bar{p} \cdot \bar{y} \text{ for all } y \in Y,$$

  and hence $\phi_Y(\bar{p}) < \bar{p} \cdot \bar{y}$, by the Separating Hyperplane Theorem.
- Since Y satisfies free disposal, i.e., $Y - \mathbb{R}^N_+ \subset Y$ (which implies $Y - \mathbb{R}^N_{++} \subset Y$), we have $\bar{p} \geq 0$ by Lemma 6.7.
- Hence, $\bar{y} \notin$ (RHS).

Proposition 6.11

If Y is nonempty, convex, and closed and satisfies free disposal, no free production, and the ability to shut down, then

$$Y = \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \gg 0\}.$$

Cost Minimization Problem

$f\colon \mathbb{R}_+^{L-1} \to \mathbb{R}_+$: production function

$$\min_{z \geq 0} \quad w \cdot z \qquad \text{(CMP)}$$
$$\text{s.t.} \quad f(z) \geq q$$

- Conditional factor demand correspondence:

$$z(w,q) = \arg\min\{w \cdot z \mid z \geq 0,\ f(z) \geq q\}$$

- Cost function:

$$c(w,z) = \min\{w \cdot z \mid z \geq 0,\ f(z) \geq q\}$$

- Analogous to expenditure minimization!

Properties of c and z

Proposition 6.12

1. *c is homogeneous of degree one in w and nondecreasing in q.*
2. *c is concave in w.*
3. *If f is continuous, nondecreasing, and quasi-concave, then $Y = \{(-z, q) \mid z \geq 0$ and $w \cdot z \geq c(w, q)$ for all $w \gg 0\}$.*
4. *z is homogeneous of degree zero in w.*
5. *If f is quasi-concave, then $z(w, q)$ is a convex set.*
   If f is continuous and strictly quasi-concave, then $z(w, q)$ is single-valued.
6. *[Shepard's lemma] If $z(w, q)$ is a singleton, then $\nabla_w c(w, q) = z(w, q)$.*
7. *If z is a continuously differentiable function, then $D_w z(w, q)$ is symmetric and negative semi-definite, and $D_w z(w, q) w = 0$.*

8. If f is homogeneous of degree one (i.e., exhibits constant returns to scale), then c and z are homogeneous of degree one in q.

9. If f is concave, then c is convex in q.

Proof

8

- Fix any w, q, and $t > 0$. (We write $c(q)$ for $c(w, q)$.)
- Take any $z \geq 0$ such that $f(z) \geq q$.
- By homogeneity, we have $f(tz) \geq tq$.
  Therefore, $c(tq) \leq w \cdot (tz) = t(w \cdot z)$, or $\frac{1}{t}c(tq) \leq w \cdot z$.
- This implies that $\frac{1}{t}c(tq) \leq c(q)$.
- Take any $z \geq 0$ such that $f(z) \geq tq$.
- By homogeneity, we have $f(\frac{1}{t}z) \geq q$.
  Therefore, $c(q) \leq w \cdot (\frac{1}{t}z) = \frac{1}{t}(w \cdot z)$, or $tc(q) \leq w \cdot z$.
- This implies that $tc(q) \leq c(tq)$.

- (We write $z(q)$ for $z(w, q)$.)

$$
\begin{aligned}
& z \in z(tq) \\
& \iff z \geq 0,\ f(z) \geq tq,\ w \cdot z = c(tq) \\
& \iff \tfrac{1}{t} z \geq 0,\ f(\tfrac{1}{t} z) \geq q,\ w \cdot (\tfrac{1}{t} z) = c(q) \\
& \iff \tfrac{1}{t} z \in z(q) \\
& \iff z \in \tfrac{1}{t} z(q)
\end{aligned}
$$

9

- Take any q, q', and $\alpha \in [0,1]$.
  Write $q'' = \alpha q + (1-\alpha)q'$.
- Let $z, z' \geq 0$ be such that $f(z) \geq q$, $f(z') \geq q'$, $c(q) = w \cdot z$, and $c(q') = w \cdot z'$.
- By concavity, $f(\alpha z + (1-\alpha)z') \geq q''$.
- Then we have

$$\begin{aligned} c(q'') \leq w \cdot (\alpha z + (1-\alpha)z') &= \alpha w \cdot z + (1-\alpha) w \cdot z' \\ &= \alpha c(q) + (1-\alpha)c(q'). \end{aligned}$$

Aggregation

- $Y_1, \ldots, Y_J$: production sets of J firms
- π_j: firm j's profit function
- y_j: firm j's supply correspondence
- Aggregate supply correspondence:

$$
\begin{aligned}
y(p) &= \sum_{j=1}^{J} y_j(p) \\
&= \{y \in \mathbb{R}^L \mid y = \textstyle\sum_j y_j \text{ for some } y_j \in y_j(p),\ j = 1, \ldots, J\}
\end{aligned}
$$

- Aggregate production set:

$$Y = \sum_{j=1}^{J} Y_j = \{y \in \mathbb{R}^L \mid \textstyle\sum_j y_j \text{ for some } y_j \in Y,\ j = 1, \ldots, J\}$$

- $\pi^*(p) = \max\{p \cdot y \mid y \in Y\}$

  $y^*(p) = \arg\max\{p \cdot y \mid y \in Y\}$

Proposition 6.13

1. $\pi^*(p) = \sum_j \pi_j(p)$
2. $y^*(p) = \sum_j y_j(p)$

Proof

1

- Take any $y \in Y$.

  Then by definition, there exist $y_j \in Y_j$, $j = 1, \dots, J$, such that $y = \sum_j y_j$.

- Then

  $$p \cdot y = p \cdot \left(\sum_j y_j\right) = \sum_j (p \cdot y_j) \leq \sum_j \pi_j(p).$$

  This implies that $\pi^*(p) \leq \sum_j \pi_j(p)$.

- Take any $y_j \in Y_j$, $j = 1, \dots, J$.

- Then

  $$\sum_j (p \cdot y_j) = p \cdot \left(\sum_j y_j\right) \leq \pi^*(p).$$

  This implies that $\sum_j \pi_j(p) \leq \pi^*(p)$.

2

- We have

$$
\begin{aligned}
& y \in y^*(p) \\
\iff & y \in Y,\ p \cdot y = \pi^*(p) \\
\iff & \exists\, y_1 \in Y_1, \dots y_J \in Y_J : y = \textstyle\sum_j y_j,\ p \cdot \sum_j y_j = \pi^*(p) \\
\iff & \exists\, y_1 \in Y_1, \dots y_J \in Y_J : y = \textstyle\sum_j y_j,\ \sum_j p \cdot y_j = \sum_j \pi_j(p) \\
\iff & \exists\, y_1 \in Y_1, \dots y_J \in Y_J : y = \textstyle\sum_j y_j,\ p \cdot y_j = \pi_j(p)\ \forall\, j \\
& (\because\ p \cdot y_j \leq \pi_j(p)\ \forall\, j = 1, \dots, J) \\
\iff & \exists\, y_1 \in y_1(p), \dots y_J \in y_J(p) : y = \textstyle\sum_j y_j \\
\iff & y \in \textstyle\sum_j y_j(p)
\end{aligned}
$$

Efficient Production

- A production vector $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.
- $y \in Y$ is *weakly efficient* if there is no $y' \in Y$ such that $y' \gg y$.
- y: efficient $\Rightarrow$ y: weakly efficient
- We say that $\bar{y} \in Y$ is profit maximizing for p if $p \cdot \bar{y} \geq p \cdot y$ for all $y \in Y$.

Proposition 6.14

1. *If $\bar{y} \in Y$ is profit maximizing for some $p \gg 0$, then $\bar{y}$ is efficient.*

2. *If $\bar{y} \in Y$ is profit maximizing for some $p \geq 0$ with $p \neq 0$, then $\bar{y}$ is weakly efficient.*

- A version of the first fundamental theorem of welfare economics

Proof

1. If $\bar{y} \in Y$ is not efficient, then there exists $y \in Y$ such that $y \geq \bar{y}$, $y \neq \bar{y}$;

   for any $p \gg 0$ we have $p \cdot (y - \bar{y}) > 0$, or $p \cdot y > p \cdot \bar{y}$.

2. If $\bar{y} \in Y$ is not weakly efficient, then there exists $y \in Y$ such that $y \gg \bar{y}$;

   for any $p \geq 0$, $p \neq 0$ we have $p \cdot (y - \bar{y}) > 0$, or $p \cdot y > p \cdot \bar{y}$.

Proposition 6.15

Suppose that Y is convex.
Any weakly efficient $\bar{y} \in Y$ is profit maximizing for some $p \geq 0$ with $p \neq 0$.

- A version of the second fundamental theorem of welfare economics

Proof

- Let $\bar{y} \in Y$ be weakly efficient.
- Then $(Y - \{\bar{y}\}) \cap \mathbb{R}^N_{++} = \emptyset$, where $Y - \{\bar{y}\}$ is convex.
- Thus by Proposition 6.8, there exists $p \geq 0$ with $p \neq 0$ such that $p \cdot z \leq 0$ for all $z \in Y - \{\bar{y}\}$, or $p \cdot y \leq p \cdot \bar{y}$ for all $y \in Y$.