5. Choice under Uncertainty

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Expected Utility Theory (von Neumann and Morgenstern)

- $C = \{x_1, \ldots, x_N\}$: (finite) set of outcomes
- $\mathcal{L} = \{(p_1, \ldots, p_N) \in \mathbb{R}^N_+ \mid p_1 + \cdots + p_N = 1\}$:
  set of alternatives ("lotteries")
- Compound lotteries

  Compound lottery of $L = (p_1, \ldots, p_N)$ and $L' = (q_1, \ldots, q_N)$:
  $\alpha L + (1 - \alpha)L' \quad (\alpha \in [0, 1])$
- We identify the compound lottery $\alpha L + (1 - \alpha)L'$ and its reduced probability distribution
  $(\alpha p_1 + (1 - \alpha)q_1, \ldots, \alpha p_N + (1 - \alpha)q_N) \in \mathcal{L}$.

- $\succsim$: complete and transitive preference relation on $\mathcal{L}$

Definition 5.1 (Continuity)

$\succsim$ on $\mathcal{L}$ satisfies continuity if for all $L, L', L'' \in \mathcal{L}$,
$\{\alpha \in [0,1] \mid \alpha L + (1-\alpha)L' \succsim L''\}$ and
$\{\alpha \in [0,1] \mid L'' \succsim \alpha L + (1-\alpha)L'\}$ are closed.

Definition 5.2 (Independence)

$\succsim$ on $\mathcal{L}$ satisfies independence if for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$,

$$L \succsim L' \iff \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

Independence Axiom

$N = 3$

- Indifference curves are straight lines:
  - If $L \sim L'$, then $\alpha L + (1-\alpha)L' \sim L'$.
  - If $L = \alpha L + (1-\alpha)L \sim L' = \alpha L'' + (1-\alpha)L$, then $L \sim L''$.
- Indifference lines are parallel to each other.

Expected Utility Theorem

Proposition 5.1

Complete and transitive $\succsim$ *on* $\mathcal{L}$ *satisfies continuity and independence if and only if there exists a function* $u\colon C \to \mathbb{R}$ *such that the function* $U\colon \mathcal{L} \to \mathbb{R}$ *defined by*

$$U(L) = \sum_{i=1}^{N} p_i u(x_i) \qquad (L = (p_1, \ldots, p_N) \in \mathcal{L})$$

represents $\succsim$*.*
Such u *is unique up to positive affine transformation (i.e., if* u *and* v *are such functions, then* $v = au + b$ *for some* $a > 0$ *and* b*).*

- Call u a von Neumann-Morgenstern (vNM) function (or Bernoulli function).

  This is not a utility function.

- U is a particular utility function that represents $\succsim$, which is of linear form.

Proof

Outline

- By finiteness of C and Independence, there are $\overline{L}, \underline{L} \in \mathcal{L}$ such that $\overline{L} \succsim L \succsim \underline{L}$ for all $L \in \mathcal{L}$. (Exercise 6.B.3)

  Assume $\overline{L} \succ \underline{L}$.

- By Continuity, for each $L \in \mathcal{L}$, there exists $\alpha \in [0, 1]$ such that

  $$L \sim \alpha \overline{L} + (1 - \alpha)\underline{L},$$

  which is unique by Independence.

- Define $U \colon \mathcal{L} \to \mathbb{R}$ by $U(L) = \alpha$.

- Verify that U represents $\succsim$.

- Verify that U is linear.

- Let $u(x_i) = U([x_i])$ (where $[x_i] \in \mathcal{L}$ is the degenerate lottery that yields $x_i \in C$ with probability one).

Step 0

- By Independence, for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$,

$$L \succ L' \Longrightarrow \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'', \tag{Ind-1}$$
$$L \sim L' \Longrightarrow \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''. \tag{Ind-2}$$

In particular,

$$L \sim L',\ L'' \sim L''' \Longrightarrow \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L'''. \tag{Ind-3}$$

Step 1 (Strict betweenness)

- By (Ind-1), for all $L, L' \in \mathcal{L}$ and $\alpha \in (0,1)$,

$$L \succ L' \Longrightarrow L \succ \alpha L + (1-\alpha)L' \succ L'. \tag{Bet}$$

(Let $L'' = L$ and $L'' = L'$ in (Ind-1).)

Step 2 (Mixture monotonicity)

- By (Bet), for all $L, L' \in \mathcal{L}$ and $\alpha, \beta \in [0,1]$, if $L \succ L'$, then

$$\alpha L + (1-\alpha)L' \succsim \beta L + (1-\beta)L' \iff \alpha \geq \beta. \quad \text{(Mon)}$$

- $\Leftarrow$: If $\alpha = \beta$, we have $\alpha L + (1-\alpha)L' \sim \beta L + (1-\beta)L'$. Suppose that $\alpha > \beta$. By (Bet), we have

$$\alpha L + (1-\alpha)L' \succ L'.$$

Since $\beta/\alpha \in (0,1)$, again by (Bet) we have

$$\alpha L+(1-\alpha)L' \succ \frac{\beta}{\alpha}\{\alpha L+(1-\alpha)L'\}+\left(1-\frac{\beta}{\alpha}\right)L' = \beta L+(1-\beta)L'.$$

- $\Rightarrow$: Contraposition.

Step 3 (Unique mixture intermediate value)

- By Continuity (with $L = \overline{L}$, $L' = \underline{L}$, and $L'' = L$), for each $L \in \mathcal{L}$, there exists $\alpha \in [0, 1]$ such that

$$L \sim \alpha \overline{L} + (1 - \alpha)\underline{L}.$$

- By (Mon), for each $L \in \mathcal{L}$, there exists a unique $\alpha \in [0, 1]$ such that

$$L \sim \alpha \overline{L} + (1 - \alpha)\underline{L}. \tag{Uni}$$

- Denote the unique such α by α_L.

Step 4 (Utility representation)

- Define $U\colon \mathcal{L} \to \mathbb{R}$ by $U(L) = \alpha_L$ for each $L \in \mathcal{L}$.
- U is a utility function that represents $\succsim$:

  For $L, L' \in \mathcal{L}$,

$$
\begin{aligned}
& L \succsim L' \\
& \iff \alpha_L \overline{L} + (1-\alpha_L)\underline{L} \succsim \alpha_{L'} \overline{L} + (1-\alpha_{L'})\underline{L} \qquad \text{(by (Uni))} \\
& \iff \alpha_L \geq \alpha_{L'} \quad \text{(by (Mon))} \\
& \iff U(L) \geq U(L').
\end{aligned}
$$

Step 5 (Linearity)

- U is linear:
  $U(\alpha L + (1-\alpha)L') = \alpha U(L) + (1-\alpha)U(L')$ for all $\alpha \in [0,1]$:

  By definition,

  $$L \sim U(L)\overline{L} + (1 - U(L))\underline{L}, \quad L' \sim U(L')\overline{L} + (1 - U(L'))\underline{L}.$$

  Therefore, by (Ind-3),

  $$\begin{aligned} \alpha L + (1-\alpha)L' &\sim \alpha\big\{U(L)\overline{L} + (1 - U(L))\underline{L}\big\} \\ &\qquad + (1-\alpha)\big\{U(L')\overline{L} + (1 - U(L'))\underline{L}\big\} \\ &= \big\{\alpha U(L) + (1-\alpha)U(L')\big\}\overline{L} \\ &\qquad + \big[1 - \big\{\alpha U(L) + (1-\alpha)U(L')\big\}\big]\underline{L}. \end{aligned}$$

  Hence,

  $$U(\alpha L + (1-\alpha)L') = \alpha U(L) + (1-\alpha)U(L').$$

Step 6 (Expected utility form)

- U has an expected utility form: there is a function $u\colon C \to \mathbb{R}$ such that $U(L) = \sum_{i=1}^{N} p_i u(x_i)$ for all $L = (p_1, \dots, p_N) \in \mathcal{L}$:

  Define u by $u(x_i) = U([x_i])$.

  Then for each $L = (p_1, \dots, p_N) \in \mathcal{L}$, which is written as $L = \sum_{i=1}^{N} p_i [x_i]$, by the linearity of U we have

$$U(L) = \sum_{i=1}^{N} p_i U([x_i]) = \sum_{i=1}^{N} p_i u(x_i).$$

Step 7 (Identification)

- Let u be as constructed above.

  By construction,

$$[x_i] \sim u(x_i)\overline{L} + (1 - u(x_i))\underline{L}.$$

- If $V(L) = \sum_{i=1}^{N} p_i v(x_i)$, then for all x_i,

$$\begin{aligned} v(x_i) &= V(u(x_i)\overline{L} + (1 - u(x_i))\underline{L}) \\ &= u(x_i)V(\overline{L}) + (1 - u(x_i))V(\underline{L}) \\ &= (V(\overline{L}) - V(\underline{L}))u(x_i) + V(\underline{L}). \end{aligned}$$

Allais Paradox

- $C = \{1\text{M}, 0.9\text{M}, 0\}$
- $L_1 = 0.9[1\text{M}] + 0.1[0] \qquad L_1' = 1[0.9\text{M}]$
- $L_2 = 0.45[1\text{M}] + 0.55[0] \qquad L_2' = 0.5[0.9\text{M}] + 0.5[0]$

Risk Aversion

- $C = \mathbb{R}_+$: set of monetary outcomes
- $\mathcal{L}$: set of cumulative distribution functions on $\mathbb{R}_+$
- Expected utility representation:

$$U(F) = \int_0^\infty u(x)dF(x) \qquad (F \in \mathcal{L})$$

  $u \colon \mathbb{R}_+ \to \mathbb{R}$: vNM function

- Assume that u is increasing and continuous.

- DM is risk averse if for any $F \in \mathcal{L}$, $1[\int x dF(x)] \succsim F$, i.e.,

$$u\left(\int x dF(x)\right) \geq \int u(x) dF(x).$$

  - $\iff$ u is concave.
- DM is risk neutral if for any $F \in \mathcal{L}$, $1[\int x dF(x)] \sim F$, i.e.,

$$u\left(\int x dF(x)\right) = \int u(x) dF(x).$$

  - $\iff$ u is affine.

- DM is strictly risk averse if for any nondegenerate $F \in \mathcal{L}$, $1[\int xdF(x)] \succ F$, i.e.,

$$u\left(\int xdF(x)\right) > \int u(x)dF(x).$$

    - $\iff$ u is strictly concave.

Certainty Equivalent, Risk Premium

- Certainty equivalent: $c(F,u)$ such that $1[c(F,u)] \sim F$, i.e., $u(C(F,u)) = \int u(x)dF(x)$
- Risk premium: $RP(F,u) = \int x dF(x) - c(F,u)$
- $RP(F,u) \geq 0$ for all F if and only if DM is risk averse.

Example: Insurance

- $(1-\pi)[w] + \pi[w-D] \qquad (\pi \in (0,1))$
- Insurance:

  costs q, pays 1 if the loss D occurs
- α units of insurance:

  $(1-\pi)[w-q\alpha] + \pi[w-q\alpha-D+\alpha]$
- Expected utility with strictly concave u:

$$f(\alpha) = (1-\pi)u(w-q\alpha) + \pi u(w-q\alpha-D+\alpha)$$

- Assume $q = \pi \cdots$ "actuarial fairness"
  (market clearing condition under free entry of insurance firms)

- $f'(\alpha) = \pi(1-\pi)(u'(w - D + (1-\pi)\alpha) - u'(w - \pi\alpha))$
- $f'(0) = \pi(1-\pi)(u'(w-D) - u'(w)) > 0$ by strict concavity
- By FOC:

$$u'(w - D + (1-\pi)\alpha^*) = u'(w - \pi\alpha^*)$$

  or

$$w - D + (1-\pi)\alpha^* = w - \pi\alpha^*$$

- Therefore, $\alpha^* = D$,

  i.e., under actuarial fairness, DM insures completely.

- In fact, using FOC is not necessary to reach this conclusion.
- If $q = \pi$, then the expected wealth is:

$$(1-\pi)(w-\pi\alpha)+\pi(w-\pi\alpha-D+\alpha)=w-\pi D$$

  for any α.
- Lottery with $\alpha = D$: $1[w-\pi D]$

  $\cdots$ preferred to any nondegenerate lottery by strictly risk averse DM

Absolute/Relative Risk Aversion

- Assume $u'(x) > 0$ for all x.
- $r_{\mathrm{A}}(x) = -\frac{u''(x)}{u'(x)}$: coefficient of absolute risk aversion at x
- $r_{\mathrm{R}}(x) = -\frac{xu''(x)}{u'(x)}$: coefficient of relative risk aversion at x

- Fix x, and consider the lottery $\frac{1}{2}[x+\varepsilon]+\frac{1}{2}[x-\varepsilon]$.
- Risk premium $RP(\varepsilon)$ satisfies

$$u(x-RP(\varepsilon))=\frac{1}{2}u(x+\varepsilon)+\frac{1}{2}u(x-\varepsilon).$$

- By second-order Taylor expansion around $\varepsilon=0$, we have

$$RP(\varepsilon)\approx\frac{1}{2}\underbrace{\left(-\frac{u''(x)}{u'(x)}\right)}_{r_{\mathrm{A}}(x)}\varepsilon^2 \qquad (\varepsilon\approx 0).$$

- Consider the lottery $\frac{1}{2}[x + \varepsilon x] + \frac{1}{2}[x - \varepsilon x]$.
- Similarly, we have

$$RP(\varepsilon) \approx \frac{1}{2}\left(-\frac{u''(x)}{u'(x)}\right)(\varepsilon x)^2,$$

or

$$\frac{RP(\varepsilon)}{x} \approx \frac{1}{2}\underbrace{\left(-\frac{xu''(x)}{u'(x)}\right)}_{r_{\mathrm{R}}(x)}\varepsilon^2.$$

Constant Absolute Risk Aversion (CARA) Functions

- $-\frac{u''(x)}{u'(x)} = a$ for all x
- $\Rightarrow u(x) = -\frac{1}{a}e^{-ax}$

  (and its positive affine transformations)

Constant Relative Risk Aversion (CRRA) Functions

- $-\frac{xu''(x)}{u'(x)} = c$ for all x
- $\Rightarrow u(x) = \begin{cases} \frac{1}{1-c}x^{1-c} & \text{if } c \neq 1 \\ \log x & \text{if } c = 1 \end{cases}$

  (and its positive affine transformations)
- (For each x, $\lim_{c\to 1} \frac{1}{1-c}(x^{1-c} - 1) = \log x$)

First-Order Stochastic Dominance

Definition 5.3

F first-order stochastically dominates G if

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

for all nondecreasing functions u.

Proposition 5.2

F first-order stochastically dominates G if and only if $F(x) \leq G(x)$ (or equivalently, $1 - F(x) \geq 1 - G(x)$) for all x.

- If $F = (p_1, p_2, p_3)$, then $\sum u(x_i)p_i = u(x_1) + (u(x_2) - u(x_1))(p_2 + p_3) + (u(x_3) - u(x_2))p_3$.