Mathematics II
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Homework 5

Due on May 15

1. Prove Lemma 7.4.

2. Suppose that $B \subset \mathbb{R}^N$, $B \neq \emptyset$, is convex and closed. For each $x \in \mathbb{R}^N$, define $f(x)$ to be the unique element $y^* \in B$ such that $\|y^* - x\| = \min_{z \in B} \|z - x\|$. Prove that f is continuous.

3. Find an example of sets $A, B \subset \mathbb{R}^N$ such that A and B are convex and closed, while there exists no $p \in \mathbb{R}^N$ such that $\sup_{x \in A} p \cdot x < \inf_{y \in B} p \cdot y$.

4. For $K \subset \mathbb{R}^N$, $K \neq \emptyset$, let $\pi_K \colon \mathbb{R}^N \to (-\infty, \infty]$ be the support function of K, i.e., the function defined by $\pi_K(p) = \sup_{x \in K} p \cdot x$, and let the correspondence $S_K \colon \mathbb{R}^N \to \mathbb{R}^N$ be defined by $S_K(p) = \{x \in \mathbb{R}^N \mid x \in K,\ \pi_K(p) = p \cdot x\}$.

For $K \neq \emptyset$, prove the following:

(1) $\pi_{\mathrm{Co}\,K}(p) = \pi_K(p)$ for all $p \in \mathbb{R}^N$.
(2) $S_{\mathrm{Co}\,K}(p) = \mathrm{Co}\,S_K(p)$ for all $p \in \mathbb{R}^N$.
(3) $\pi_{\mathrm{Cl}\,K}(p) = \pi_K(p)$ for all $p \in \mathbb{R}^N$.

5. For $K \subset \mathbb{R}^N$, $K \neq \emptyset$, let $\pi_K \colon \mathbb{R}^N \to (-\infty, \infty]$ be the support function of K, i.e., the function defined by $\pi_K(p) = \sup_{x \in K} p \cdot x$. Show that if $K \neq \emptyset$ is a cone, then for each $p \in \mathbb{R}^N$, either $\pi_K(p) = 0$ or $\pi_K(p) = \infty$.

6. Prove the following:

Suppose that $K \subset \mathbb{R}^N$, $K \neq \emptyset$, is a cone. For $p \in \mathbb{R}^N$, if there exists $c \in \mathbb{R}$ such that $p \cdot x \geq c$ for all $x \in K$, then $\inf_{x \in K} p \cdot x = 0$.

7. For $Y \subset \mathbb{R}^N$, $Y \neq \emptyset$, denote

$$Y' = \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \in \mathbb{R}^N_+\},$$
$$Y'' = \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \in \mathbb{R}^N_{++}\},$$

where $\phi_Y \colon \mathbb{R}^N \to (-\infty, \infty]$ is the support function of Y, i.e., the function defined by $\phi_Y(p) = \sup_{y \in Y} p \cdot y$.

Assume that Y is convex and closed and satisfies free disposal.

(1) Give an example of Y for which $Y' \neq Y''$.
(2) Prove the following:
If $\phi_Y(p) < \infty$ for all $p \in \mathbb{R}^N_{++}$, then $Y' = Y''$.

8. Prove Proposition 7.14.

9. For $A \in \mathbb{R}^{M \times N}$, prove that $\{A^{\mathrm{T}}x \in \mathbb{R}^N \mid x \in \mathbb{R}^M_+\}$ is a closed set by using Farkas' Lemma.

10. Prove Farkas' Lemma by using the inequality version of Farkas' Lemma (Proposition 7.18).