

Homework 6

Due on May 22

1. We want to prove “2 \Rightarrow 1” in the inequality version of Farkas’ Lemma (Proposition 7.18) by using Strong Duality for Linear Programming (Proposition 7.21).

Let $A \in \mathbb{R}^{M \times N}$ and $b \in \mathbb{R}^N$ be given, and assume that Condition 2 in Proposition 7.18 holds, i.e., that for any $y \in \mathbb{R}^N$, if $y \geq 0$ and $Ay \geq 0$, then $b^T y \geq 0$.

Consider the linear program

$$(P) \quad \begin{aligned} \max_{x \in \mathbb{R}^M} \quad & 0^T x \\ \text{s. t.} \quad & A^T x \leq b \\ & x \geq 0. \end{aligned}$$

- (1) Write down the dual problem of (P).
- (2) Show that the dual problem has a solution.
- (3) Use Proposition 7.21 to conclude that Condition 1 in Proposition 7.18 holds.

2. Prove the following:

Let $K \subset \mathbb{R}^N$, $K \neq \emptyset$, be a compact convex set. If $K \cap \mathbb{R}_+^N = \emptyset$, then there exist $p \gg 0$ and $c < 0$ such that

$$p \cdot x \leq c \text{ for all } x \in K.$$

(*Hint.* Consider either the set $A = K - \mathbb{R}_+^N$ or the set $B = \text{Co}(K \cup (-\Delta))$ (where $\Delta = \{x = (x_1, \dots, x_N) \in \mathbb{R}_+^N \mid x_1 + \dots + x_N = 1\}$), each of which is closed by the compactness of K .)

3. Prove Ville’s Theorem (Proposition 7.24) using the proposition proved in Problem 2.

4. For $A \in \mathbb{R}^{M \times N}$, let

$$\begin{aligned} v^*(A) &= \max_{x \in \Delta_M} \min_{y \in \Delta_N} x^T Ay, \\ v^{**}(A) &= \min_{y \in \Delta_N} \max_{x \in \Delta_M} x^T Ay, \end{aligned}$$

where $\Delta_K = \{z = (z_1, \dots, z_K) \in \mathbb{R}_+^K \mid z_1 + \dots + z_K = 1\}$.

(1) Prove that for every $A \in \mathbb{R}^{M \times N}$, $v^*(A) \leq v^{**}(A)$.

(2) By using Ville's Theorem (Proposition 7.24), prove that for every $A \in \mathbb{R}^{M \times N}$, either $v^*(A) > 0$ holds or $v^{**}(A) \leq 0$ holds.

(3) By using (2), prove that for every $A \in \mathbb{R}^{M \times N}$, $v^*(A) = v^{**}(A)$.

(Hint: Prove by contradiction by assuming that $v^*(A) < v^{**}(A)$.)

5. Prove Motzkin's Theorem (Proposition 7.28) by using Farkas' Lemma.

6. For $A \subset \mathbb{R}^N$, $x \in A$ is called an *extreme point* of A if there do not exist $y, z \in A$ and $\alpha \in (0, 1)$ such that $y \neq z$ and $x = (1 - \alpha)y + \alpha z$.

Let $B = \{x^1, \dots, x^m\} \subset \mathbb{R}^N$ (finite subset of \mathbb{R}^N), and let $C = \text{Co } B$ (convex hull of B). Suppose that $\bar{x} \in C$ is an extreme point of C .

(1) Prove the following:

There exists $i_0 \in \{1, \dots, m\}$ such that $\bar{x} = x^{i_0}$ and $\bar{x} \notin \text{Co}(B \setminus \{x^{i_0}\})$.

(2) Prove the following:

There exists $p \in \mathbb{R}^N$ such that $p \cdot \bar{x} > p \cdot x$ for all $x \in C \setminus \{\bar{x}\}$.

(3) Consider the following statement:

For any closed convex set $D \subset \mathbb{R}^N$, if $\bar{x} \in D$ is an extreme point of D , then there exists $p \in \mathbb{R}^N$ such that $p \cdot \bar{x} > p \cdot x$ for all $x \in D \setminus \{\bar{x}\}$.

Determine whether this statement is true or false, and provide a proof if true or provide a counter-example if false.

7. Determine the local maximizers and minimizers of the following functions.

(1) $f(x_1, x_2) = x_1^3 + x_2^3 - x_1^2 + x_1x_2 - x_2^2$

(2) $f(x_1, x_2) = x_1^4 - 4x_1x_2 + 2x_2^2$

8. Given $p_1, p_2, w > 0$, consider the utility maximization problem:

$$\max_{x_1, x_2} u(x_1, x_2) = x_1 + \log(x_2 + 1)$$

$$\text{s. t. } p_1x_1 + p_2x_2 \leq w$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

(1) Show that u (as a function defined on \mathbb{R}_+^2) is quasi-concave.

(2) Write down the KKT conditions.

(3) Derive the Walrasian demand function.

9. Prove Proposition 8.9.