10. Fixed Point Theorems

Daisuke Oyama

Mathematics II

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Brouwer's Fixed Point Theorem

Proposition 10.1 (Brouwer's Fixed Point Theorem)

Suppose that $X \subset \mathbb{R}^N$ *is a nonempty, compact, and convex set, and that* $f\colon X \to X$ *is a continuous function from* X *into itself. Then* f *has a fixed point, i.e., there exists* $x \in X$ *such that* $x = f(x)$*.*

- What if X is not compact?
- What if X is not convex?
- What if f is not continuous?

Kakutani's Fixed Point Theorem

Proposition 10.2 (Kakutani's Fixed Point Theorem)

Suppose that $X \subset \mathbb{R}^N$ is a nonempty, compact, and convex set, and that $F\colon X \to X$ is a correspondence from X into itself that is

1. *nonempty-valued,*
2. *convex-valued,*
3. *compact-valued, and*
4. *upper semi-continuous.*

Then F has a fixed point, i.e., there exists $x \in X$ such that $x \in F(x)$.

Note:

Since the codomain is compact,
"being compact-valued and upper semi-continuous" can be replaced with "having a closed graph".

- What if F is not convex-valued?
- What if F is not compact-valued?

Proof of Brouwer's Fixed Point Theorem

- Sperner's Lemma
- KKM Lemma
- Brouwer's Fixed Point Theorem
  - for simplices
  - for general compact convex sets

Simplices

- The *unit simplex* Δ in $\mathbb{R}^N$ is the set

$$\begin{aligned}\Delta &= \left\{x \in \mathbb{R}^N \mid x_1, \ldots, x_N \geq 0,\ \textstyle\sum_{i=1}^N x_i = 1\right\} \\ &= \mathrm{Co}\{e_1, \ldots, e_N\},\end{aligned}$$

  where $e_i \in \mathbb{R}^N$ is the ith unit vector in $\mathbb{R}^N$.

- An m-*simplex* in $\mathbb{R}^N$ is the convex hull $\mathrm{Co}\{a^1, a^2, \ldots, a^{m+1}\}$ of $m+1$ affinely independent vectors $a^1, a^2, \ldots, a^{m+1}$ in $\mathbb{R}^N$

  (i.e., $a^2 - a^1, \ldots, a^{m+1} - a^1$ linearly independent).

  - $\Delta \subset \mathbb{R}^N$ is an $(N-1)$-simplex in $\mathbb{R}^N$.

- For an m-simplex $S = \mathrm{Co}\{a^1, \ldots, a^{m+1}\}$:
  - Each a^i is called a *vertex* of the m-simplex,

    where we write $V(S) = \{a^1, \ldots, a^{m+1}\}$ (set of vertices of S).
  - $\{a^1, \ldots, a^{m+1}\}$ is said to *span* S.
  - The simplex spanned by a subset of vertices of S is called a *face* of S,

    where a face spanned by k vertices is called a k-face.
  - For each $x \in S$, which is (uniquely) represented by a convex combination $\sum_i \alpha_i a^i$, the *carrier* $C(x)$ of x is the set of indices with positive weights:

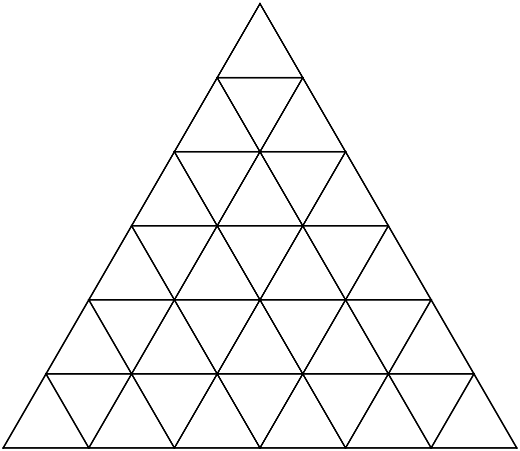
$$C(x) = \{i \in \{1, \ldots, m+1\} \mid \alpha_i > 0\}.$$

Simplicial Subdivision

- A *simplicial subdivision* of an m-simplex S is a finite set of m-simplices (subsimplices) such that
  - the union of all subsimplices is S, and
  - the intersection of any two subsimplices is either empty or a face of both.
- The *mesh* of a simplicial subdivision is the maximum among the diameters of the subsimplices.

  (The diameter of a set A is $\sup_{x,y \in A} \|x - y\|$.)
  - For any $\varepsilon > 0$, there exists a simplicial subdivision with mesh smaller than ε.

Example: Equilateral subdivision



Sperner Labelling

- Consider a simplicial subdivision $\mathcal{T}$ of an m-simplex $S = \mathrm{Co}\{a^1, \dots, a^{m+1}\}$.

  Let $V(\mathcal{T})$ be the set of vertices of subsimplices in $\mathcal{T}$.

- A *Sperner labelling* (or *proper labelling*) of $\mathcal{T}$ is a mapping $\lambda\colon V(\mathcal{T}) \to \{1, \dots, m+1\}$ such that

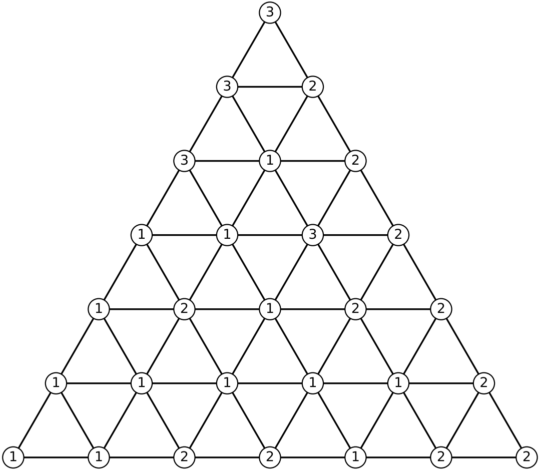
  $$\lambda(v) \in C(v)$$

  for all $v \in V(\mathcal{T})$

  (where $C(v) = \{i \in \{1, \dots, m+1\} \mid \alpha_i > 0\}$ is the carrier of v).

- A subsimplex in $\mathcal{T}$ is *completely labelled* if its set of vertices has all $m+1$ distinct labels.

Example: Sperner labelling



Sperner's Lemma

Proposition 10.3 (Sperner's Lemma)

For any simplicial subdivision of any m-simplex and any Sperner labelling of it,
there are an odd number of completely labelled subsimplices;
in particular, there is at least one completely labelled subsimplex.

Proof

By induction in m:

- The statement is trivial for $m = 0$.
- For $m \geq 1$, assume that the statement is true for $m - 1$.
- Let a simplicial subdivision $\mathcal{T}$ of an m-simplex S and a Sperner labelling $\lambda\colon \mathcal{T} \to \{1, \ldots, m+1\}$ be given.
- Define
  - C: set of subsimplices with labels $\{1, \ldots, m+1\}$ (set of completely labelled subsimplices);
  - A: set of subsimplices with labels $\{1, \ldots, m\}$ (set of "almost" completely labelled subsimplices); and
  - E: set of $(m-1)$-faces of subsimplices with labels $\{1, \ldots, m\}$ that are contained in the boundary of S.

- Let $R = C \cup A \cup E$.
- Define

$$\mathcal{D} = \{(t,t') \in R \times R \mid t \neq t',\ \lambda(V(t \cap t')) = \{1,\dots,m\}\}.$$

($V(t \cap t')$: set of vertices of the simplex $t \cap t'$)

- Interpretation
  - S: house; $\mathcal{T}$: rooms
  - C: rooms with labels $\{1,\dots,m+1\}$
  - A: rooms with labels $\{1,\dots,m\}$
  - E: entrances (outside of the house)
  - $\mathcal{D}$: doors between two rooms or a room and an entrance
    $\cdots$ defined as ordered pairs, hence each door counted twice

1. For each $t \in C$: $|\{t' \mid (t, t') \in \mathcal{D}\}| = 1$.

   "Each room in C has one door."

2. For each $t \in A$: $|\{t' \mid (t, t') \in \mathcal{D}\}| = 2$.

   ($\because$ One label in $\{1, \ldots, m\}$ is repeated.)

   "Each room in A has two doors."

3. For each $t \in E$: $|\{t' \mid (t, t') \in \mathcal{D}\}| = 1$.

   "Each entrance in E has one door."

4. $|\mathcal{D}|$: even

   ($\because$ $(t, t') \in \mathcal{D} \iff (t', t) \in \mathcal{D}$)

   "Each door in $\mathcal{D}$ is counted twice."

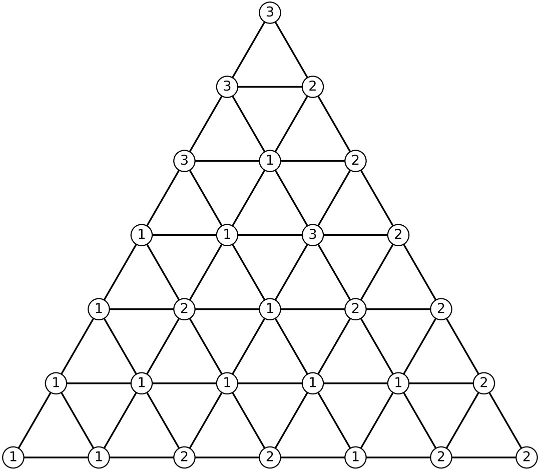
- Therefore, we have

$$|\mathcal{D}| = |C| + 2|A| + |E|,$$

  where $|\mathcal{D}|$ is even.

- Therefore, $|C| + |E|$ is even.

- Since $|E|$ is odd by the induction hypothesis, it therefore follows that $|C|$ is odd.


3
3 2
3 1 2
1 1 3 2
1 2 1 2 2
1 1 1 1 1 2
1 1 2 2 1 2 2

Paths through doors

- 4 types of paths:
  - $e \leftrightarrow \cdots \leftrightarrow a \leftrightarrow \cdots \leftrightarrow e$
  - $e \leftrightarrow \cdots \leftrightarrow a \leftrightarrow \cdots \leftrightarrow c$
  - $c \leftrightarrow \cdots \leftrightarrow a \leftrightarrow \cdots \leftrightarrow c$
  - $\cdots \leftrightarrow a \leftrightarrow \cdots$ (cycle)

  where $c \in C$, $a \in A$, $e \in E$.
- $|E|$ is odd by the induction hypothesis.

KKM (Knaster-Kuratowski-Mazurkiewicz) Lemma

Let $\Delta = \mathrm{Co}\{e_1, \ldots, e_N\}$ be the unit simplex in $\mathbb{R}^N$, where $e_i \in \mathbb{R}^N$ is the ith unit vector in $\mathbb{R}^N$.

Proposition 10.4 (KKM Lemma)

Let $F_1, \ldots, F_N$ be a family of closed subsets of Δ such that

$$\mathrm{Co}\{e_i \mid i \in I\} \subset \bigcup_{i \in I} F_i \text{ for every } I \subset \{1, \ldots, N\}. \qquad (*)$$

Then we have $\bigcap_{i=1}^N F_i \neq \emptyset$.

Proof

- Let $F_1, \ldots, F_N$ be a family of closed subsets of Δ that satisfy condition $(*)$.
- For each $k \in \mathbb{N}$, let $\mathcal{T}_k$ be a simplicial subdivision of Δ with mesh smaller than $\frac{1}{k}$, and $V(\mathcal{T}_k)$ the set of vertices of subsimplices in $\mathcal{T}_k$.
- For each $v \in V(\mathcal{T}_k)$, where $v \in \mathrm{Co}\{e_i \mid i \in C(v)\}$, by condition $(*)$ there is some $i \in C(v)$ such that $v \in F_i$.

  Let $\lambda_k(v)$ be any such i.
- Then, the mapping $\lambda_k \colon V(\mathcal{T}_k) \to \{1, \ldots, N\}$ so defined is a Sperner labelling.

- Therefore, by Sperner's Lemma, there exists a completely labelled subsimplex in $\mathcal{T}_k$.

  Denote its vertices by $v^1(k), \ldots, v^N(k)$ so that $\lambda_k(v^i(k)) = i$.
- By construction, $v^i(k) \in F_i$ for all k.
- By the compactness of Δ, $\{v^1(k)\}_{k=1}^{\infty}$ has a convergent subsequence $\{v^1(k_n)\}_{n=1}^{\infty}$ with a limit x^*.
- Since the diameter of $\mathrm{Co}\{v^1(k_n), \ldots, v^N(k_n)\}$ converges to 0 as $n \to \infty$, we have $v^i(k_n) \to x^*$ also for all $i \neq 1$.
- By the closedness of F_i, we have $x^* \in F_i$ for all $i = 1, \ldots, N$.

Brouwer's Fixed Point Theorem for Simplices

Proposition 10.5

If $f : \Delta \to \Delta$ is continuous, then it has a fixed point.

Corollary 10.6

For a simplex S,
if $f : S \to S$ is continuous, then it has a fixed point.

Proof of Brouwer's Fixed Point Theorem for Unit Simplex

- Let $f\colon \Delta \to \Delta$ be continuous, where we write $f(x) = (f_1(x), \ldots, f_N(x))$.
- For each $i \in \{1, \ldots, N\}$, define a subset F_i of Δ by

$$F_i = \{x \in \Delta \mid x_i \geq f_i(x)\},$$

which is closed by the continuity of f.

- If $x \in \bigcap_{i=1}^N F_i$, which means $x_i \geq f_i(x)$ for all i, then we have

$$1 = \sum_{i=1}^N x_i \geq \sum_{i=1}^N f_i(x) = 1,$$

and hence $x_i = f_i(x)$ for all i, i.e., x is a fixed point of f.

- Therefore, it suffices to show that $\bigcap_{i=1}^N F_i \neq \emptyset$.

- For any $I \subset \{1, \ldots, N\}$,
  if $x \in \mathrm{Co}\{e_i \mid i \in I\}$, then $x \in \bigcup_{i \in I} F_i$.
  - ∵ If $x \notin \bigcup_{i \in I} F_i$, which means $x_i < f_i(x)$ for all $i \in I$, then we would have

$$1 = \sum_{i=1}^{N} x_i = \sum_{i \in I} x_i < \sum_{i \in I} f_i(x) \leq \sum_{i=1}^{N} f_i(x) = 1,$$

    which is a contradiction.

- Thus, $F_1, \ldots, F_N$ satisfy the hypothesis of the KKM Lemma.
- Therefore, by the KKM Lemma, $\bigcap_{i=1}^{N} F_i \neq \emptyset$, as desired.

Proof of Brouwer's Fixed Point Theorem

- Let X be a nonempty, compact, and convex set, and $f\colon X \to X$ continuous.
- Let S be a sufficiently large simplex that contains X.
- For each $x \in S$, let $g(x)$ be the unique $y \in X$ such that $\|y - x\| = \inf_{z \in X} \|z - x\|$.

  The function $g\colon S \to X$ is well defined and continuous by the closedness and convexity of X.
- Define $h\colon S \to S$ by $h(x) = f(g(x))$, which is continuous.
- By Corollary 10.6, h has a fixed point $x^* \in S$, which must be in X.
- Then, we have $x^* = h(x^*) = f(g(x^*)) = f(x^*)$, i.e., x^* is a fixed point of f.

Proof of Kakutani's Fixed Point Theorem for Simplices

- Let $S \subset \mathbb{R}^N$ be an M-simplex: $S = \mathrm{Co}\{a^1, \dots, a^{M+1}\}$.
- Let $F\colon S \to S$ be a nonempty- and convex valued correspondence from S to S whose graph is closed.
- For each $k \in \mathbb{N}$, let $\mathcal{T}_k$ be a simplicial subdivision of S with mesh smaller than $\frac{1}{k}$, and $V(\mathcal{T}_k)$ the set of vertices of subsimplices in $\mathcal{T}_k$.
- For each k, we construct a continuous function f^k from S to S as follows:
  - For each $v \in V(\mathcal{T}_k)$, take any $y \in F(v)$, and let $f^k(v) = y$.
  - For each $x \in S$,

    if x is in a subsimplex $\mathrm{Co}\{v^1, \dots, v^{M+1}\}$, so that $x = \sum_{m=1}^{M+1} \alpha_m v^m$,

    then let $f^k(x) = \sum_{m=1}^{M+1} \alpha_m y^m$, where $y^m = f^k(v^m)$.

- By Brouwer's Fixed Point Theorem, f^k has a fixed point $x^k \in S$: $f^k(x^k) = x^k$.
- Write $x^k = \sum_{m=1}^{M+1} \alpha_m^k v^{k,m}$ and $f^k(x^k) = \sum_{m=1}^{M+1} \alpha_m^k y^{k,m}$, where $y^{k,m} = f^k(v^{k,m}) \in F(v^{k,m})$.
- By taking a subsequence, as $k \to \infty$,

  $x^k \to x^*$, $\alpha_m^k \to \alpha_m^*$, and $y^{k,m} \to y^{*,m}$,

  and also, $v^{k,m} \to x^*$.
- From $x^k = f^k(x^k) = \sum_{m=1}^{M+1} \alpha_m^k y^{k,m}$, we have $x^* = \sum_{m=1}^{M+1} \alpha_m^* y^{*,m}$.
- From $y^{k,m} \in F(v^{k,m})$, we have $y^{*,m} \in F(x^*)$ by the closedness of the graph of F.
- Therefore, by the convexity of $F(x^*)$, we have $x^* \in F(x^*)$.

Application: Existence of Nash Equilibrium

Nash gave three proofs of the existence of Nash equilibrium of finite normal form games.

1. J. F. Nash, "Equilibrium Points in n-Person Games," *Proceedings of the National Academy of Sciences of the United States of America* 36 (1950), 48-49.
2. J. F. Nash, "Non-Cooperative Games," Dissertation, Princeton University, Department of Mathematics, 1950.
3. J. Nash, "Non-Cooperative Games," *Annals of Mathematics* 54 (1951), 287-295.

See also:

- J. Hofbauer, "From Nash and Brown to Maynard Smith: Equilibria, Dynamics and ESS," Selection 1 (2000), 81-88.

Normal Form Games

Definition 10.1

An I-player (finite) normal form game is a tuple $(\mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}})$ where

- $\mathcal{I} = \{1, \ldots, I\}$ is the set of players,
- S_i is the finite set of strategies of player $i \in \mathcal{I}$, and
- $u_i \colon \prod_j S_j \to \mathbb{R}$ is the payoff function of player $i \in \mathcal{I}$.

Mixed Strategies (1/2)

- A mixed strategy σ_i of player i is a probability distribution over S_i, where $\sigma_i(s_i)$ denotes the probability that i plays $s_i \in S_i$.
- We denote by $\Delta(S_i)$ the set of mixed strategies of player i.
- $\Delta(S_i)$ is a convex and compact subset of $\mathbb{R}^{|S_i|}$.
- $\prod_i \Delta(S_i)$ is a convex and compact subset of $\mathbb{R}^{|S_1|+\cdots+|S_I|}$.

Mixed Strategies (2/2)

- For $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_I) \in \prod_{j \neq i} \Delta(S_j)$, we write

$$u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \prod_{j \neq i} \sigma_j(s_j) u_i(s_i, s_{-i}),$$
$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}).$$

- $u_i(s_i, \sigma_{-i})$ is continuous in σ_{-i}.
- $u_i(\sigma_i, \sigma_{-i})$ is continuous in (σ_i, σ_{-i}).
- $u_i(\sigma_i, \sigma_{-i})$ is linear in σ_i.

Nash Equilibrium (in Mixed Strategies)

Definition 10.2

A mixed strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_I^*) \in \prod_i \Delta(S_i)$ is a *Nash equilibrium* of $(\mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}})$ if for all $i \in \mathcal{I}$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Delta(S_i)$.

Equivalent Representations

1. Define the correspondences $B_i \colon \prod_{j \neq i} \Delta(S_j) \to \Delta(S_i)$ and $B \colon \prod_j \Delta(S_j) \to \prod_j \Delta(S_j)$ by

   $$B_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) \mid u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \; \forall \, \sigma_i' \in \Delta(S_i)\},$$
   $$B(\sigma) = B_1(\sigma_{-1}) \times \cdots \times B_I(\sigma_{-I}).$$

   σ^* is a Nash equilibrium if and only if σ^* is a fixed point of B, i.e., $\sigma^* \in B(\sigma^*)$.

2. σ^* is a Nash equilibrium if and only if for all $i \in \mathcal{I}$ and $s_i \in S_i$,

   $$\sigma_i^*(s_i) > 0 \Rightarrow u_i(s_i, \sigma_{-i}^*) = \max_{s_i' \in S_i} u_i(s_i', \sigma_{-i}^*).$$

3. σ^* is a Nash equilibrium if and only if for all $i \in \mathcal{I}$,

   $$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \text{ for all } s_i \in S_i.$$

Existence Theorem

Proposition 10.7

Every finite normal form game has at least one Nash equilibrium.

Three Proofs

1. J. F. Nash, "Equilibrium Points in n-Person Games," *Proceedings of the National Academy of Sciences of the United States of America* 36 (1950), 48-49.
2. J. F. Nash, "Non-Cooperative Games," Dissertation, Princeton University, Department of Mathematics, 1950.
3. J. Nash, "Non-Cooperative Games," *Annals of Mathematics* 54 (1951), 287-295.

First Proof (1/2)

- B is a correspondence from the nonempty, convex, and compact set $\prod_j \Delta(S_j)$ to itself.
- $B_i(\sigma_{-i}) \subset \mathbb{R}^{|S_i|}$ is the set of all convex combinations of pure best responses to σ_{-i}, which is nonempty and convex.

  So B is nonempty- and convex-valued.
- To show that B has a closed graph,
  let $(\sigma^k, \tau^k) \in \prod_j \Delta(S_j) \times \prod_j \Delta(S_j)$ be such that $\tau_i^k \in B_i(\sigma_{-i}^k)$ for each i, and suppose that $(\sigma^k, \tau^k) \to (\sigma, \tau)$ as $k \to \infty$.

First Proof (2/2)

- Take any i and any σ'_i. Then $u_i(\tau_i^k, \sigma_{-i}^k) \geq u_i(\sigma'_i, \sigma_{-i}^k)$.

  Since u_i is continuous, letting $k \to \infty$ we have

  $$u_i(\tau_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}).$$

  This means $\tau_i \in B_i(\sigma_{-i})$.

- Therefore, all the conditions of Kakutani's Fixed Point Theorem are satisfied.
- Hence, B has a fixed point, which is a Nash equilibrium.

Second Proof (1/4)

- For each $i \in \mathcal{I}$ and for $k \in \mathbb{N}$, define the function $b_i^k \colon \prod_{j \neq i} \Delta(S_j) \to \Delta(S_i)$ by

$$b_i^k(\sigma_{-i})(s_i) = \frac{\phi_{is_i}^k(\sigma_{-i})}{\sum_{s_i' \in S_i} \phi_{is_i'}^k(\sigma_{-i})},$$

  where

$$\phi_{is_i}^k(\sigma_{-i}) = \left[u_i(s_i, \sigma_{-i}) - U_i(\sigma_{-i}) + \frac{1}{k} \right]_+,$$

  and $U_i(\sigma_{-i}) = \max_{s_i' \in S_i} u_i(s_i', \sigma_{-i})$ and $[x]_+ = \max\{x, 0\}$.

- $b_i^k(\sigma_{-i})(s_i) > 0$ if and only if $u_i(s_i, \sigma_{-i}) > U_i(\sigma_{-i}) - \frac{1}{k}$.
  "Play $\frac{1}{k}$-best responses with positive probabilities."

- b_i^k is a continuous function.

Second Proof (2/4)

- Define the function $b^k\colon \prod_j \Delta(S_j) \to \prod_j \Delta(S_j)$ by

$$b^k(\sigma) = (b_1^k(\sigma_{-1}), \ldots, b_I^k(\sigma_{-I})).$$

- b^k is a continuous function from the nonempty, convex, and compact set $\prod_j \Delta(S_j)$ to itself.
- Therefore, by Brouwer's Fixed Point Theorem
b^k has a fixed point, i.e., there exists $\sigma^k \in \prod_j \Delta(S_j)$
such that $\sigma^k = b^k(\sigma^k)$.
- Since $\prod_j \Delta(S_j)$ is a compact set, the sequence $\{\sigma^k\}$ has a convergent subsequence with a limit $\sigma^* \in \prod_j \Delta(S_j)$.

  We want to show that σ^* is a Nash equilibrium.
- Take any $i \in \mathcal{I}$ and any $s_i \in S_i$ such that $\sigma_i^*(s_i) > 0$.

  Fix any $\varepsilon > 0$.

Second Proof (3/4)

- Since $\sigma_i^k \to \sigma_i^*$ and $U_i(\cdot) - u_i(s_i, \cdot)$ is continuous, we can take a k such that
  - $\sigma_i^k(s_i) > 0 \left(\iff u_i(s_i, \sigma_{-i}^k) - U_i(\sigma_{-i}^k) + \frac{1}{k} > 0 \right)$,
  - $[U_i(\sigma_{-i}^*) - u_i(s_i, \sigma_{-i}^*)] - [U_i(\sigma_{-i}^k) - u_i(s_i, \sigma_{-i}^k)] < \frac{\varepsilon}{2}$, and
  - $\frac{1}{k} < \frac{\varepsilon}{2}$.
- Therefore,

$$
\begin{aligned}
0 \le U_i(\sigma_{-i}^*) - u_i(s_i, \sigma_{-i}^*) \\
&= \left([U_i(\sigma_{-i}^*) - u_i(s_i, \sigma_{-i}^*)] - [U_i(\sigma_{-i}^k) - u_i(s_i, \sigma_{-i}^k)] \right) \\
&\quad + \left(U_i(\sigma_{-i}^k) - u_i(s_i, \sigma_{-i}^k) - \frac{1}{k} \right) + \frac{1}{k} \\
&< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Second Proof (4/4)

- So we have shown that $u_i(s_i, \sigma^*_{-i}) = U_i(\sigma^*_{-i})$ whenever $\sigma_i^*(s_i) > 0$.
- This means that σ^* is a Nash equilibrium.

Third Proof (1/3)

- For each $i \in \mathcal{I}$, define the function $f_i \colon \prod_j \Delta(S_j) \to \Delta(S_i)$ by

$$f_i(\sigma)(s_i) = \frac{\sigma_i(s_i) + k_{is_i}(\sigma)}{1 + \sum_{s_i' \in S_i} k_{is_i'}(\sigma)},$$

where

$$k_{is_i}(\sigma) = \left[u_i(s_i, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i})\right]_+ .$$

- f_i is a continuous function.
- Define the function $f \colon \prod_j \Delta(S_j) \to \prod_j \Delta(S_j)$ by

$$f(\sigma) = (f_1(\sigma), \ldots, f_I(\sigma)).$$

- f is a continuous function from the nonempty, convex, and compact set $\prod_j \Delta(S_j)$ to itself.

Third Proof (2/3)

- Therefore, by Brouwer's Fixed Point Theorem f has a fixed point, i.e., there exists $\sigma^* \in \prod_j \Delta(S_j)$ such that for all $i \in \mathcal{I}$ and $s_i \in S_i$,

$$\sigma_i^*(s_i) = \frac{\sigma_i^*(s_i) + k_{is_i}(\sigma^*)}{1 + \sum_{s_i' \in S_i} k_{is_i'}(\sigma^*)},$$

  hence $\sigma_i^*(s_i) \sum_{s_i' \in S_i} k_{is_i'}(\sigma^*) = k_{is_i}(\sigma^*)$, where

$$k_{is_i}(\sigma^*) = \left[u_i(s_i, \sigma_{-i}^*) - u_i(\sigma_i^*, \sigma_{-i}^*)\right]_+ .$$

- We want to show that σ^* is a Nash equilibrium.

Third Proof (3/3)

- By the linearity of u_i in σ_i, there is some $\bar{s}_i$ with $\sigma_i^*(\bar{s}_i) > 0$ such that

$$u_i(\bar{s}_i, \sigma_{-i}^*) \leq u_i(\sigma_i^*, \sigma_{-i}^*),$$

  for which we have $k_{i\bar{s}_i}(\sigma^*) = 0$.

- But by $\sigma_i^*(\bar{s}_i) \sum_{s_i' \in S_i} k_{is_i'}(\sigma^*) = k_{i\bar{s}_i}(\sigma^*)$, we have

$$\sum_{s_i' \in S_i} k_{is_i'}(\sigma^*) = 0,$$

  and hence, $k_{is_i}(\sigma^*) = 0$ for all $s_i \in S_i$.

- That is, we have $u_i(s_i, \sigma_{-i}^*) \leq u_i(\sigma_i^*, \sigma_{-i}^*)$ for all $s_i \in S_i$.
- This implies that σ^* is a Nash equilibrium.

Tarski's Fixed Point Theorem

Let X be any nonempty set.

- For functions $v\colon X \to \mathbb{R}$ and $v'\colon X \to \mathbb{R}$, we write $v \leq v'$ if $v(x) \leq v'(x)$ for all $x \in X$.
- This order $\leq$ defines a partial order on the set of functions from X to $\mathbb{R}$.
- Fix two functions $\underline{v}\colon X \to \mathbb{R}$ and $\overline{v}\colon X \to \mathbb{R}$ such that $\underline{v} \leq \overline{v}$, and write

$$[\underline{v}, \overline{v}] = \{v\colon X \to \mathbb{R} \mid \underline{v} \leq v \leq \overline{v}\}.$$

- A function $\varphi\colon [\underline{v}, \overline{v}] \to [\underline{v}, \overline{v}]$ is nondecreasing if for all $v, v' \in [\underline{v}, \overline{v}]$, $v \leq v' \Rightarrow \varphi(v) \leq \varphi(v')$.

Tarski's Fixed Point Theorem

Proposition 10.8 (Tarski's Fixed Point Theorem)

Suppose that $\varphi\colon [\underline{v}, \overline{v}] \to [\underline{v}, \overline{v}]$ *is nondecreasing.*
Then φ *has a fixed point, i.e., there exists* $v^* \in [\underline{v}, \overline{v}]$ *such that* $v^* = \varphi(v^*)$.

Proof (1/3)

- Let

$$A = \{v \in [\underline{v}, \overline{v}] \mid v \leq \varphi(v)\}$$

(which is nonempty since $\underline{v} \in A$).

- Define the function $v^*\colon X \to \mathbb{R}$ by

$$v^*(x) = \sup\{v(x) \mid v \in A\}$$

for each $x \in X$ (which is well defined since $\{v(x) \mid v \in A\}$ is bounded above by $\overline{v}(x)$ and hence its supremum exists).

- Clearly, $v^* \in [\underline{v}, \overline{v}]$.
- Note that v^* is the least upper bound of A, that is, if $v \leq u$ for all $v \in A$, then $v^* \leq u$.
- We want to show that v^* is a fixed point of φ.

Proof (2/3)

- Fix any $v \in A$. Thus, $v \leq \varphi(v)$ by the definition of A.
- By the definition of v^*, $v \leq v^*$, and thus $\varphi(v) \leq \varphi(v^*)$ by the assumption that φ is nondecreasing.
- Therefore, we have $v \leq \varphi(v^*)$.
- Since this holds for any $v \in A$, it means that $\varphi(v^*)$ is an upper bound of A.
- Hence,

$$v^* \leq \varphi(v^*) \tag{1}$$

  since v^* is the least upper bound of A.
- Again by the assumption that φ is nondecreasing, it follows from (1) that $\varphi(v^*) \leq \varphi(\varphi(v^*))$, and hence $\varphi(v^*) \in A$.

Proof (3/3)

- Hence,

$$\varphi(v^*) \leq v^* \tag{2}$$

 by the definition of v^*.

- Therefore, by (1) and (2), we have $v^* = \varphi(v^*)$.

Contraction Mapping Fixed Point Theorem

Let X be any nonempty set.

- Let $\mathcal{B}(X)$ be the set of bounded functions from X to $\mathbb{R}$.
- Define the function $d \colon \mathcal{B}(X) \times \mathcal{B}(X) \to \mathbb{R}_+$ by

$$d(v, v') = \sup_{x \in X} |v(x) - v'(x)| \qquad (v, v' \in \mathcal{B}(X)).$$

- d satisfies the following properties:
  1. $d(v, v') = 0$ if and only if $v = v'$;
  2. $d(v, v') = d(v', v)$;
  3. $d(v, v') \leq d(v, v'') + d(v'', v')$.
- A function $\varphi \colon \mathcal{B}(X) \to \mathcal{B}(X)$ is a *contraction mapping* (or simply, contraction) if there exists $\beta \in (0, 1)$ such that

$$d(\varphi(v), \varphi(v')) \leq \beta d(v, v')$$

  for all $v, v' \in \mathcal{B}(X)$.

Contraction Mapping Fixed Point Theorem

Proposition 10.9 (Contraction Mapping Fixed Point Theorem)

Suppose that $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ *is a contraction mapping.*
Then φ *has a unique fixed point, i.e., there exists a unique* $v^* \in \mathcal{B}(X)$ *such that* $v^* = \varphi(v^*)$.
Moreover, for any $v^0 \in \mathcal{B}(X)$, $d(\varphi^m(v^0), v^*) \to 0$ *as* $m \to \infty$, *where* $\varphi^m(v^0) = \varphi(\varphi^{m-1}(v^0))$, $m = 1, 2, \ldots$.

Proof (1/3)

- Fix any $v^0 \in \mathcal{B}(X)$, and consider the sequence $\{v^m\}$ defined by $v^m = \varphi(v^{m-1})$ for $m \in \mathbb{N}$.
- Then the sequence $\{v^m\}$ is a *Cauchy sequence* in $\mathcal{B}(X)$ in the following sense:

  for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that

  $$d(v^m, v^n) < \varepsilon$$

  for all $m, n \geq M$.

  ($\because$ Given $\varepsilon > 0$, let $M \in \mathbb{N}$ be such that $[\beta^M/(1-\beta)]d(\varphi(v^0), v^0) < \varepsilon$.)
- Then for each $x \in X$,
  the sequence $\{v^m(x)\}$ is a Cauchy sequence in $\mathbb{R}$, and hence it converges to some real number by the completeness of $\mathbb{R}$.

  Denote the limit by $v^*(x)$.

Proof (2/3)

- Regarding the function $v^*\colon X \to \mathbb{R}$ so defined, one can show:
  1. $v^* \in \mathcal{B}(X)$, i.e., v^* is bounded;
  2. $d(v^m, v^*) \to 0$ as $m \to \infty$.
- We show that v^* is indeed a fixed point of φ.
- Fix any $\varepsilon > 0$. Let $M \in \mathbb{N}$ be such that $d(v^m, v^*) < \varepsilon/(1+\beta)$ for all $m \geq M$.

  Then we have

$$
\begin{aligned}
d(\varphi(v^*), v^*) &\leq d(\varphi(v^*), \varphi(v^M)) + d(\varphi(v^M), v^*) \\
&\leq \beta d(v^*, v^M) + d(v^{M+1}, v^*) < \varepsilon.
\end{aligned}
$$

- Since $\varepsilon > 0$ has been taken arbitrarily, it follows that $d(\varphi(v^*), v^*) = 0$ and hence $\varphi(v^*) = v^*$.

Proof (3/3)

- Uniqueness:

  Let $\varphi(v^*) = v^*$ and $\varphi(v^{**}) = v^{**}$.

  Then

  $$d(v^*, v^{**}) = d(\varphi(v^*), \varphi(v^{**})) \leq \beta d(v^*, v^{**}),$$

  and therefore $(1 - \beta)d(v^*, v^{**}) \leq 0$.

  Since $\beta < 1$, we have $d(v^*, v^{**}) \leq 0$, and therefore $v^* = v^{**}$.

- Convergence:

  We have shown that for any choice of $v^0 \in \mathcal{B}(X)$,
  the sequence $\{v^m\}$ defined by $v^m = \varphi(v^{m-1})$ for $m \in \mathbb{N}$
  converges to the unique fixed point v^*.

Remark

- The only property of $\mathcal{B}(X)$ (and d) used in the proof is its *completeness*,

  i.e., the property that any Cauchy sequence in the set converges to some element of that set.

- For example, one can show that for $X \subset \mathbb{R}^N$, the set $\mathcal{C}_b(X)$ of bounded and *continuous* functions from X to $\mathbb{R}$ in fact satisfies this property.

  Therefore, the Contraction Mapping Theorem holds also with $\mathcal{C}_b(X)$ in place of $\mathcal{B}(X)$ (with the same d).