Mathematics II
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Homework 1

Due on April 15

1. Prove Proposition 1.1.

2. Let $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Show that A has no least upper bound in $\mathbb{Q}$. (You can use the fact that $\sqrt{2} \notin \mathbb{Q}$.)

3. Prove the following:

For any $x \in \mathbb{R}$, there exists $m \in \mathbb{Z}$ such that $m - 1 \leq x < m$.

4. Prove the following:

(1) $2^{-m} < \dfrac{1}{m}$ for all $m \in \mathbb{N}$.
(2) For $a > 0$, $\lim_{m\to\infty} 2^{-m} a = 0$.

5. Prove the following:

(1) For sequences $\{x^m\}_{m=1}^\infty$ and $\{y^m\}_{m=1}^\infty$ in $\mathbb{R}$, if $x^m \leq y^m$ for all $m \in \mathbb{N}$, $\lim_{m\to\infty} x^m = \alpha$, and $\lim_{m\to\infty} y^m = \beta$, then $\alpha \leq \beta$.
(2) For sequences $\{x^m\}_{m=1}^\infty$, $\{y^m\}_{m=1}^\infty$, and $\{z^m\}_{m=1}^\infty$ in $\mathbb{R}$, if $x^m \leq z^m \leq y^m$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} x^m = \lim_{m\to\infty} y^m = \alpha$, then $\lim_{m\to\infty} z^m = \alpha$.

6. Let $\{x^m\}_{m=1}^\infty$ and $\{y^m\}_{m=1}^\infty$ be sequences in $\mathbb{R}$, and suppose that $\lim_{m\to\infty} x^m = \alpha$ and $\lim_{m\to\infty} y^m = \beta$.

(1) Define $\{z^m\}_{m=1}^\infty$ by $z^m = x^m + y^m$ for $m \in \mathbb{N}$. Prove that $\lim_{m\to\infty} z^m = \alpha + \beta$.
(2) Define $\{w^m\}_{m=1}^\infty$ by $w^m = x^m y^m$ for $m \in \mathbb{N}$. Prove that $\lim_{m\to\infty} w^m = \alpha\beta$.

7. Prove the following:

(1) A convergent sequence $\{x^m\}_{m=1}^\infty$ in $\mathbb{R}$ is a Cauchy sequence.
(2) A Cauchy sequence $\{x^m\}_{m=1}^\infty$ in $\mathbb{R}$ is bounded (i.e., there exists $M \geq 0$ such that $|x^m| \leq M$ for all $m \in \mathbb{N}$).

8. Prove Proposition 1.7.

9. Show that the Archimedean Property and the Completeness of $\mathbb{R}$ imply the Axiom of Real Numbers.

Hint. Let $A \subset \mathbb{R}$ be nonempty and bounded above, and consider the sets $B = \{x \in \mathbb{R} \mid a \leq x \text{ for all } a \in A\}$ and $C = \mathbb{R} \setminus B$, both of which are nonempty (why?).

10. For $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, let us denote

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_N|\},$$
$$\|x\|_1 = |x_1| + \cdots + |x_N|.$$

(1) Show that $\|x\|_\infty \leq \|x\| \leq \|x\|_1 \leq N\|x\|_\infty$ for any $x \in \mathbb{R}^N$.

(2) Show that for a sequence $\{x^m\}_{m=1}^\infty$ in $\mathbb{R}^N$, the following conditions are equivalent:

(i) $x^m \to \bar{x}$ as $m \to \infty$ (i.e., $\|x^m - \bar{x}\| \to 0$ as $m \to \infty$).

(ii) $\|x^m - \bar{x}\|_\infty \to 0$ as $m \to \infty$.

(iii) For each $i = 1, \dots, N$, $x_i^m \to \bar{x}_i$ as $m \to \infty$.

(iv) $\|x^m - \bar{x}\|_1 \to 0$ as $m \to \infty$.

11. Let $\{x^m\}$ be a sequence in $\mathbb{R}^N$, and $\bar{x} \in \mathbb{R}^N$.

(1) Prove the following:

If there exist $\varepsilon > 0$ and $M \in \mathbb{N}$ such that $\|x^m - \bar{x}\| \geq \varepsilon$ for all $m \geq M$, then no subsequence of $\{x^m\}$ converges to $\bar{x}$.

(2) Prove that the following statements are equivalent:

(i) $\{x^m\}$ converges to $\bar{x}$.

(ii) Every subsequence of $\{x^m\}$ converges to $\bar{x}$.

(iii) Every subsequence of $\{x^m\}$ has a subsequence that converges to $\bar{x}$.

12.

(1) Prove Proposition 2.3.

(2) Prove Proposition 2.4.

13. Prove the following.

(1) For $x \in \mathbb{R}^N$ and $\varepsilon > 0$, $B_\varepsilon(x)$ is an open set.

(2) For $A \subset \mathbb{R}^N$ and $\varepsilon > 0$, $B_\varepsilon(A)$ $(= \{x \in \mathbb{R}^N \mid \|y - x\| < \varepsilon \text{ for some } y \in A\})$ is an open set.

14. For $X \subset \mathbb{R}^N$ and $A \subset X$, prove the following:

(1) $\operatorname{Cl}_X A = (\operatorname{Cl} A) \cap X$.

(2) A is closed relative to X if and only if $\operatorname{Bdry}_X A \subset A$.

15. What are the interior, the closure, and the boundary of $\mathbb{Q} \cap [0, 1]$ (relative to $\mathbb{R}$)?