Mathematics II
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Homework 3

Due on April 30

1. Prove Proposition 4.1.

2. Prove Proposition 4.2.

3. Prove Proposition 4.3.

4. Prove Proposition 4.5.

5. Prove (LHS) $\subset$ (RHS) in Proposition 4.6.

6. Prove Proposition 4.16.

7. Prove the following:

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a concave function. If $x < x'$ and $t > 0$, then

$$f(x+t) - f(x) \geq f(x'+t) - f(x').$$

8. For a function $f\colon \mathbb{R}^N \to \mathbb{R}$, consider the following conditions:

(i) $f(\alpha x) = |\alpha| f(x)$ for all $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$.
(ii) $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}^N$.
(iii) If $f(x+y) = f(x) + f(y)$ and $x \neq 0$, then $y = \alpha x$ for some $\alpha \geq 0$.

Prove the following:

(1) If f satisfies (i) and (ii), then it is a convex function.
(2) If f satisfies (i), (ii), and (iii), then $f(x) > 0$ whenever $x \neq 0$.
(3) If f satisfies (i), (ii), and (iii), then it is a strictly quasi-convex function.

9. Suppose that $f\colon \mathbb{R}^N \to [-\infty, \infty]$ is a concave function. Show that for any $c \in [-\infty, \infty]$, the sets $\{x \in \mathbb{R}^N \mid f(x) > c\}$ and $\{x \in \mathbb{R}^N \mid f(x) \geq c\}$ are convex.

($f\colon \mathbb{R}^N \to [-\infty, \infty]$ is defined to be concave if $\operatorname{hyp} f = \{(x, y) \in \mathbb{R}^N \times \mathbb{R} \mid y \leq f(x)\}$ is convex.)

10.

(1) For $x \in \mathbb{R}^N$ and $\varepsilon > 0$, show that $B_\varepsilon(x)$ is a convex set.
(2) For a convex set $C \subset \mathbb{R}^N$ and $\varepsilon > 0$, show that $B_\varepsilon(C)$ $(= \{x \in \mathbb{R}^N \mid \|x - y\| < \varepsilon \text{ for some } y \in C\})$ is a convex set.

11. Prove Proposition 4.22.

12. Show that $f\colon \mathbb{R}^N \to [-\infty, \infty]$ is concave if and only if

$$f((1-\alpha)x + \alpha x') > (1-\alpha)t + \alpha t'$$

whenever $f(x) > t$, $f(x') > t'$, and $0 \leq \alpha \leq 1$.

13. Let $X \subset \mathbb{R}^N$ and $A \subset \mathbb{R}^S$ be nonempty convex sets. For a function $f\colon X \times A \to \mathbb{R}$, consider the function $v\colon A \to [-\infty, \infty]$ be defined by

$$v(\alpha) = \sup_{x \in X} f(x, \alpha).$$

Show that if f is concave, then v is concave.

14. Prove Proposition 4.24.

15. Prove Proposition 4.26.