Mathematics II
Daisuke Oyama
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Homework 5

Due on May 13

1. Prove Lemma 7.4.

2. Suppose that $B \subset \mathbb{R}^N$, $B \neq \emptyset$, is convex and closed. For each $x \in \mathbb{R}^N$, define $f(x)$ to be the unique element $y^* \in B$ such that $\|y^* - x\| = \min_{z \in B} \|z - x\|$. Prove that f is continuous.

3. Find an example of sets $A, B \subset \mathbb{R}^N$ such that A and B are convex and closed, and $A \cap B = \emptyset$, while there exists no $p \in \mathbb{R}^N$ such that $\sup_{x \in A} p \cdot x < \inf_{y \in B} p \cdot y$.

4. For $K \subset \mathbb{R}^N$, $K \neq \emptyset$, let $\pi_K \colon \mathbb{R}^N \to (-\infty, \infty]$ be the support function of K, i.e., the function defined by $\pi_K(p) = \sup_{x \in K} p \cdot x$, and let the correspondence $S_K \colon \mathbb{R}^N \to \mathbb{R}^N$ be defined by $S_K(p) = \{x \in \mathbb{R}^N \mid x \in K,\ \pi_K(p) = p \cdot x\}$.

For $K \neq \emptyset$, prove the following:

(1) $\pi_{\mathrm{Co}\, K}(p) = \pi_K(p)$ for all $p \in \mathbb{R}^N$.

(2) $S_{\mathrm{Co}\, K}(p) = \mathrm{Co}\, S_K(p)$ for all $p \in \mathbb{R}^N$.

(3) $\pi_{\mathrm{Cl}\, K}(p) = \pi_K(p)$ for all $p \in \mathbb{R}^N$.

5. For $K \subset \mathbb{R}^N$, $K \neq \emptyset$, let $\pi_K \colon \mathbb{R}^N \to (-\infty, \infty]$ be the support function of K, i.e., the function defined by $\pi_K(p) = \sup_{x \in K} p \cdot x$. Show that if $K \neq \emptyset$ is a cone, then for each $p \in \mathbb{R}^N$, either $\pi_K(p) = 0$ or $\pi_K(p) = \infty$.

6. Prove the following:

Suppose that $K \subset \mathbb{R}^N$, $K \neq \emptyset$, is a cone. For $p \in \mathbb{R}^N$, if there exists $c \in \mathbb{R}$ such that $p \cdot x \geq c$ for all $x \in K$, then $\inf_{x \in K} p \cdot x = 0$.

7. For $Y \subset \mathbb{R}^N$, $Y \neq \emptyset$, denote

$$Y' = \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \in \mathbb{R}^N_+\},$$
$$Y'' = \{y \in \mathbb{R}^N \mid p \cdot y \leq \phi_Y(p) \text{ for all } p \in \mathbb{R}^N_{++}\},$$

where $\phi_Y \colon \mathbb{R}^N \to (-\infty, \infty]$ is the support function of Y, i.e., the function defined by $\phi_Y(p) = \sup_{y \in Y} p \cdot y$.

Assume that Y is convex and closed and satisfies free disposal.

(1) Give an example of Y for which $Y' \neq Y''$.

(2) Prove the following:

If $\phi_Y(p) < \infty$ for all $p \in \mathbb{R}^N_{++}$, then $Y' = Y''$.

8. Prove Proposition 7.14.

9. For a function $f\colon \mathbb{R}^N \to \mathbb{R}$, define the function f^* by

$$f^*(p) = \sup_{x \in \mathbb{R}^N} p \cdot x - f(x) \qquad (p \in \mathbb{R}^N),$$

where we assume that any "sup" that shows up is alway finite, so that this function f^* is a function from $\mathbb{R}^N$ to $\mathbb{R}$. Write $f^{**} = (f^*)^*$ (where again we assume that any "sup" is finite, so that this is again a function from $\mathbb{R}^N$ to $\mathbb{R}$).

(1) Suppose that $N = 1$ and $f(x) = \frac{1}{\alpha}|x|^\alpha$, where $\alpha > 1$. Compute f^*.

(2) Prove that for any function $f\colon \mathbb{R}^N \to \mathbb{R}$, f^* is a convex function.

(3) Prove that if f is a convex function, then $f^{**} = f$.

Hint. You *may* follow the following steps:

(a) Show that $f^{**}(x) \leq f(x)$ for all x, by the definition of the "$*$" operator.

(b) Assume that $f^{**}(\bar{x}) < f(\bar{x})$ for some $\bar{x}$, and obtain a contradiction, by applying the separating hyperplane theorem to the set $\operatorname{epi} f = \{(x, y) \in \mathbb{R}^{N+1} \mid y \geq f(x)\}$. (You may use the fact that the convex function f is a continuous function without proof.)

10. For $A \in \mathbb{R}^{M \times N}$, prove that $\{A^{\mathrm{T}} x \in \mathbb{R}^N \mid x \in \mathbb{R}^M_+\}$ is a closed set by using Farkas' Lemma.

11. Prove Farkas' Lemma by using the inequality version of Farkas' Lemma (Proposition 7.18).