Homework 6

Due on May 20

1. We want to prove "2 $\Rightarrow$ 1" in the inequality version of Farkas' Lemma (Proposition 7.18) by using Strong Duality for Linear Programming (Proposition 7.21).

Let $A \in \mathbb{R}^{M\times N}$ and $b \in \mathbb{R}^N$ be given, and assume that Condition 2 in Proposition 7.18 holds, i.e., that for any $y \in \mathbb{R}^N$, if $y \geq 0$ and $Ay \geq 0$, then $b^{\mathrm{T}} y \geq 0$.

Consider the linear program

$$
\text{(P)} \quad \begin{aligned} \max_{x\in\mathbb{R}^M} \quad & 0^{\mathrm{T}} x \\ \text{s.t.} \quad & A^{\mathrm{T}} x \leq b \\ & x \geq 0. \end{aligned}
$$

(1) Write down the dual problem of (P).

(2) Show that the dual problem has a solution.

(3) Use Proposition 7.21 to conclude that Condition 1 in Proposition 7.18 holds.

2. Prove the following:

Let $K \subset \mathbb{R}^N$, $K \neq \emptyset$, be a compact convex set. If $K \cap \mathbb{R}^N_+ = \emptyset$, then there exist $p \gg 0$ and $c < 0$ such that

$$p \cdot x \leq c \text{ for all } x \in K.$$

(*Hint*. Consider either the set $A = K - \mathbb{R}^N_+$ or the set $B = \mathrm{Co}(K \cup (-\Delta))$ (where $\Delta = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N_+ \mid x_1 + \cdots + x_N = 1\}$), each of which is closed by the compactness of K.)

3. Prove Ville's Theorem (Proposition 7.24) using the proposition proved in Problem **2**.

4. For $A \in \mathbb{R}^{M\times N}$, let

$$
\begin{aligned}
v^*(A) &= \max_{x\in\Delta_M} \min_{y\in\Delta_N} x^{\mathrm{T}} A y, \\
v^{**}(A) &= \min_{y\in\Delta_N} \max_{x\in\Delta_M} x^{\mathrm{T}} A y,
\end{aligned}
$$

where $\Delta_K = \{z = (z_1, \ldots, z_K) \in \mathbb{R}^K_+ \mid z_1 + \cdots + z_K = 1\}$.

(1) Prove that for every $A \in \mathbb{R}^{M\times N}$, $v^*(A) \leq v^{**}(A)$.

(2) By using Ville's Theorem (Proposition 7.24), prove that for every $A \in \mathbb{R}^{M \times N}$, either $v^*(A) > 0$ holds or $v^{**}(A) \leq 0$ holds.

(3) By using (2), prove that for every $A \in \mathbb{R}^{M \times N}$, $v^*(A) = v^{**}(A)$.

(Hint: Prove by contradiction by assuming that $v^*(A) < v^{**}(A)$.)

5. Prove Motzkin's Theorem (Proposition 7.28) by using Farkas' Lemma.

6. For $A \subset \mathbb{R}^N$, $x \in A$ is called an *extreme point* of A if there do not exist $y, z \in A$ and $\alpha \in (0,1)$ such that $y \neq z$ and $x = (1-\alpha)y + \alpha z$.

Let $B = \{x^1, \dots, x^m\} \subset \mathbb{R}^N$ (finite subset of $\mathbb{R}^N$), and let $C = \operatorname{Co} B$ (convex hull of B). Suppose that $\bar{x} \in C$ is an extreme point of C.

(1) Prove the following:

There exists $i_0 \in \{1, \dots, m\}$ such that $\bar{x} = x^{i_0}$ and $\bar{x} \notin \operatorname{Co}(B \setminus \{x^{i_0}\})$.

(2) Prove the following:

There exists $p \in \mathbb{R}^N$ such that $p \cdot \bar{x} > p \cdot x$ for all $x \in C \setminus \{\bar{x}\}$.

(3) Consider the following statement:

For any closed convex set $D \subset \mathbb{R}^N$, if $\bar{x} \in D$ is an extreme point of D, then there exists $p \in \mathbb{R}^N$ such that $p \cdot \bar{x} > p \cdot x$ for all $x \in D \setminus \{\bar{x}\}$.

Determine whether this statement is true or false, and provide a proof if true or provide a counter-example if false.

7. Determine the local maximizers and minimizers of the following functions.

(1) $f(x_1, x_2) = x_1^3 + x_2^3 - x_1^2 + x_1 x_2 - x_2^2$

(2) $f(x_1, x_2) = x_1^4 - 4x_1 x_2 + 2x_2^2$

8. Given $p_1, p_2, w > 0$, consider the utility maximization problem:

$$\begin{aligned} \max_{x_1, x_2} \quad & u(x_1, x_2) = x_1 + \log(x_2 + 1) \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 \leq w \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

(1) Show that u (as a function defined on $\mathbb{R}^2_+$) is quasi-concave.

(2) Write down the KKT conditions.

(3) Derive the Walrasian demand function.

9. Prove Proposition 8.9.