Mathematics II
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Homework 7

Due on May 27

1. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,\alpha) = -\frac{1}{4}x^4 - \frac{\alpha}{3}x^3 + \frac{1}{2}x^2 + \alpha x - \frac{1}{4}.$$

Compute the value function $v(\alpha) = \max_{x\in\mathbb{R}} f(x,\alpha)$, $\alpha \in \mathbb{R}$, and draw its graph.

2. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,\alpha) = \begin{cases} -\dfrac{1}{\alpha^3}x^2(x-2\alpha)^2 & \text{if } 2\alpha < x < 0, \\ \dfrac{1}{\alpha^3}x^2(x-2\alpha)^2 & \text{if } 0 < x < 2\alpha, \\ -x^2(x-2\alpha)^2 & \text{otherwise.} \end{cases}$$

Compute the value function $v(\alpha) = \max_{x\in\mathbb{R}} f(x,\alpha)$, $\alpha \in \mathbb{R}$, and draw its graph.

3. Let $Y \subset X \subset \mathbb{R}$. For a function $F\colon X \times X \to \mathbb{R}$ and $\beta \in \mathbb{R}$, suppose that a function $v\colon X \to \mathbb{R}$ satisfies

$$v(x) = \max_{y\in Y} F(x,y) + \beta v(y)$$

for all $x \in X$. Let $\bar{x} \in \operatorname{Int} X$ and $\bar{y} \in \arg\max_{y\in Y} F(\bar{x},y) + \beta v(y)$. Assume that (1) $F(\cdot,\bar{y})$ is differentiable at $\bar{x}$, and (2) v *is differentiable at* $\bar{x}$. Under these assumptions, derive the envelope formula:

$$v'(\bar{x}) = \frac{\partial F}{\partial x}(\bar{x},\bar{y}).$$

4.

(1) Suppose that a real valued function $w\colon \mathbb{R} \to \mathbb{R}$ is concave. Prove that for all $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$\frac{w(\alpha) - w(\alpha-\varepsilon)}{\varepsilon} \geq \frac{w(\alpha+\varepsilon) - w(\alpha)}{\varepsilon}.$$

(2) For a function $f\colon \mathbb{R}^K \times \mathbb{R} \to [-\infty,\infty]$, let the function $v\colon \mathbb{R} \to [-\infty,\infty]$ be defined by

$$v(\alpha) = \sup_{x\in\mathbb{R}^K} f(x,\alpha).$$

Prove the following:

(a) If for all $x \in \mathbb{R}^K$, $f(x, \alpha)$ is convex in α, then v is convex.

(b) If $f(x, \alpha)$ is concave in (x, α), then v is concave.

(c) Assume that $X^*(\alpha) = \{x \in \mathbb{R}^K \mid f(x, \alpha) = v(\alpha)\} \neq \emptyset$ for all $\alpha \in \mathbb{R}$, that for all $x \in \mathbb{R}^K$, $f(x, \alpha)$ is differentiable in α, and that v is concave. Then v is differentiable at any $\bar{\alpha} \in \mathbb{R}$ with

$$v'(\bar{\alpha}) = \frac{\partial f}{\partial \alpha}(\bar{x}, \bar{\alpha})$$

for any $\bar{x} \in X^*(\bar{\alpha})$.

5. Let $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ be a (row-)*stochastic matrix*, i.e., $a_{ij} \geq 0$ for all $i, j = 1, \ldots, N$ and $\sum_{j=1}^N a_{ij} = 1$ for all $i = 1, \ldots, N$. A vector $x^* \in \Delta$ is called a *stationary distribution* of A if $(x^*)^{\mathrm{T}} A = (x^*)^{\mathrm{T}}$, where $\Delta = \{x \in \mathbb{R}_+ \mid \sum_{i=1}^N x_i = 1\}$.

Show that A has a stationary distribution

(1) by using the Perron-Frobenius Theorem (Proposition 6.10),

(2) by using Farkas' Lemma, and

(3) by using Brouwer's Fixed Point Theorem.

6.

(1) Find an example of a correspondence $F \colon X \to X$ such that $X \subset \mathbb{R}^N$ is nonempty, compact, and convex, and F is nonempty- and compact-valued and upper semi-continuous, but not convex-valued, and has no fixed point.

(2) Find an example of a correspondence $F \colon X \to X$ such that $X \subset \mathbb{R}^N$ is nonempty, compact, and convex, and F is nonempty- and convex-valued and upper semi-continuous, but not compact-valued, and has no fixed point.

7. Let $X \subset \mathbb{R}^N$, and let $\{f^m\}_{m=1}^\infty$ be a sequence of bounded and continuous functions $f^m \colon X \to \mathbb{R}$ (i.e., for each m, f^m is continuous, and there exists some $r^m \in \mathbb{R}$ such that $|f^m(x)| < r^m$ for all $x \in X$). Assume that $\{f^m\}_{m=1}^\infty$ satisfies the following property:

For any $\varepsilon > 0$, there exists M such that $\sup_{x \in X} |f^m(x) - f^n(x)| < \varepsilon$ for all $m, n \geq M$.

(1) Show that for each $x \in X$, the sequence $\{f^m(x)\}_{m=1}^\infty$ of real numbers is a Cauchy sequence.

(2) By the above and the completeness of $\mathbb{R}$, for each $x \in X$, $\{f^m(x)\}_{m=1}^\infty$ is convergent; denote its limit by $f(x) \in \mathbb{R}$.

Show that the function $f \colon X \to \mathbb{R}$ is bounded.

(You *may* (but do not have to) follow the following approach:

- Let $M_1 \in \mathbb{N}$ be such that for all $x \in X$, $|f^m(x) - f^n(x)| < 1$ for all $m, n \geq M_1$.
- For each $x \in X$, let $m(x) \in \mathbb{N}$ be such that $m(x) \geq M_1$ and $|f^{m(x)}(x) - f(x)| < 1$.
- Let $r = r^{M_1} + 2$ (where $r^{M_1} \in \mathbb{R}$ is such that $|f^{M_1}(x)| < r^{M_1}$ for all $x \in X$).

Then show that for all $x \in X$, $|f(x)| < r$ (by triangular inequality).)

(3) Show that $\sup_{x \in X} |f^m(x) - f(x)| \to 0$ as $m \to \infty$.

(4) Show that f is continuous.