3. Correspondences

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Correspondences

Let X and Y be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively.

- A *correspondence* $F\colon X \to Y$ is a rule that assigns a set $F(x) \subset Y$ to every $x \in X$.
  - "$F\colon X \to\to Y$", "$F\colon X \rightrightarrows Y$", and "$F\colon X \Rightarrow Y$" are also used.
- F is *nonempty-valued* if $F(x) \neq \emptyset$ for all $x \in X$.
  - In Debreu, a correspondence is defined to be a nonempty-valued correspondence.
- F is *compact-valued* if $F(x)$ is compact for all $x \in X$.
- F is *convex-valued* if $F(x)$ is convex for all $x \in X$.
- F is *closed-valued* if $F(x)$ is closed (relative to Y) for all $x \in X$.
- F is *singleton-valued* if $F(x)$ is a singleton set for all $x \in X$.

- The *graph* of F is the set

$$\mathrm{Graph}(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

- F is *locally bounded* (or *uniformly bounded*) *near* $x \in X$ if there exists $\varepsilon > 0$ such that $F(B_\varepsilon(x) \cap X)$ is bounded.

  F is locally bounded if for all $x \in X$, it is locally bounded near x.

- $F(A) = \{y \in Y \mid y \in F(x) \text{ for some } x \in A\} = \bigcup_{x \in A} F(x)$

  $\cdots$ the *image* of A under F.

Examples

- Define $B\colon \mathbb{R}^N_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N_+$ by

$$B(p, w) = \{x \in \mathbb{R}^N_+ \mid p \cdot x \le w\}.$$

  B is a nonempty- and compact-valued correspondence.

- Given a function $u\colon \mathbb{R}^N_+ \to \mathbb{R}$,
  define the correspondence $x\colon \mathbb{R}^N_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N_+$ by

$$\begin{aligned} x(p, w) = \{x \in \mathbb{R}^N_+ \mid x \in B(p, w) &\text{ and} \\ u(x) \ge u(y) &\text{ for all } y \in B(p, w)\} \end{aligned}$$

  (the Walrasian demand correspondence).

  If u is continuous, then x is

  - nonempty-valued by the Extreme Value Theorem, and
  - compact-valued. —Why?

Continuous Correspondences: Notice

- Terminology:
  We use "upper/lower semi-continuous"
  instead of "upper/lower hemi-continuous".
- Definition:
  We adopt general definitions using open sets.
  - For lower semi-continuity,
    our definition is equivalent to that in MWG.
  - For upper semi-continuity, under some additional assumption
    our definition is equivalent to that in MWG.

Continuous Functions: Review

- For a *function* $f\colon X \to Y$, the following conditions are equivalent:

  1. For any open neighborhood V of $f(\bar{x})$ (relative to Y), there exists an open neighborhood U of $\bar{x}$ (relative to X) such that $f(U) \subset V$.
  2. For any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x}$ as $m \to \infty$, we have $f(x^m) \to f(\bar{x})$ as $m \to \infty$.

- For *correspondences*, these are no longer equivalent.

  1. Condition 1 will be used to define *upper semi-continuity*.
  2. (A generalized version of) Condition 2 will be equivalent to *lower semi-continuity*.

- 1. An upper semi-continuous correspondence
    - may have a “downward jump”, but
    - may not have an “upward jump”.
  2. A lower semi-continuous correspondence
    - may have an “upward jump”, but
    - may not have a “downward jump”.

Upper Semi-Continuity

Let X and Y be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively.

Definition 3.1

- A correspondence $F\colon X \to Y$ is upper semi-continuous at $\bar{x} \in X$ if
  for any open neighborhood V of $F(\bar{x})$ (relative to Y),
  there exists an open neighborhood U of $\bar{x}$ (relative to X)
  such that $F(U) \subset V$.

- For $A \subset X$, $F\colon X \to Y$ is upper semi-continuous on A if it is upper semi-continuous at all $\bar{x} \in A$.

- $F\colon X \to Y$ is upper semi-continuous if it is upper semi-continuous on X.

- $F(U) = \{y \in Y \mid y \in F(x) \text{ for some } x \in U\}$

  $\cdots$ the *image* of U under F.

Constant Correspondences

- Any correspondence F with $F(x) = F(x')$ for all $x, x' \in X$ is upper semi-continuous according to our definition.

Upper Semi-Continuity + Compact-Valuedness

Proposition 3.1

$F\colon X \to Y$ is upper semi-continuous at $\bar{x}$ and $F(\bar{x})$ is compact
if and only if
for any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x}$,
any sequence $\{y^m\} \subset Y$ such that $y^m \in F(x^m)$ for all $m \in \mathbb{N}$ has a convergent subsequence whose limit is in $F(\bar{x})$.

Proposition 3.2

If $F\colon X \to Y$ is upper semi-continuous and compact-valued, then $F(A)$ is compact for any compact set $A \subset X$.

- $F(A) = \{y \in Y \mid y \in F(x) \text{ for some } x \in A\}$

  $\cdots$ the *image* of A under F.

Proof of Proposition 3.1

- "Only if" part:

  Assume the contrary, i.e., that there exist $\{x^m\} \subset X$ with $x^m \to \bar{x}$ and $\{y^m\} \subset Y$ with $y^m \in F(x^m)$ for all m such that for any $z \in F(\bar{x})$, no subsequence of $\{y^m\}$ converges to z.

- Then for each $z \in F(\bar{x})$, there exists $\varepsilon(z) > 0$ such that $\{m \in \mathbb{N} \mid y^m \in B_{\varepsilon(z)}(z)\}$ is a finite set.

- By the compactness of $F(\bar{x})$, there are finitely many points $z^1, \ldots, z^K \in F(\bar{x})$ such that $F(\bar{x}) \subset \bigcup_{k=1}^{K} B_{\varepsilon(z^k)}(z^k)$ (Proposition 2.12).

- By the upper semi-continuity of F at $\bar{x}$, there exists an open neighborhood U of $\bar{x}$ such that $F(U) \subset \bigcup_{k=1}^{K} B_{\varepsilon(z^k)}(z^k)$.

- By $x^m \to \bar{x}$, there exists M such that $x^m \in U$ for all $m \geq M$, and hence $y^m \in \bigcup_{k=1}^{K} B_{\varepsilon(z^k)}(z^k)$ for all $m \geq M$.
- But this contradicts the finiteness of $\{m \in \mathbb{N} \mid y^m \in B_{\varepsilon(z^k)}(z^k)\}$ for all $k = 1, \ldots, K$.

- "If" part:

  Compactness of $F(\bar{x})$ is immediate.

- If F is not upper semi-continuous at $\bar{x}$, then there exists an open neighborhood V of $F(\bar{x})$ such that for each m, there exists x^m and y^m such that $x^m \in B_{\frac{1}{m}}(\bar{x})$, $y^m \in F(x^m)$, and $y^m \notin V$.

- Then $x^m \to \bar{x}$, while no subsequence of $\{y^m\}$ can converge to a point in $F(\bar{x})$.

Closed Graph

Definition 3.2

$F\colon X \to Y$ has a closed graph if its graph,

$$\mathrm{Graph}(F) = \{(x, y) \in X \times Y \mid y \in F(x)\},$$

is closed (relative to $X \times Y$).

Definition 3.3

- $F\colon X \to Y$ is closed at $\bar{x}$ if

$$x^m \to \bar{x},\ y^m \in F(x^m) \text{ for all } m \in \mathbb{N}, \text{ and } y^m \to y$$
$$\Rightarrow y \in F(\bar{x}).$$

- $F\colon X \to Y$ is closed if it is closed at all $\bar{x} \in X$.

Proposition 3.3

$F\colon X \to Y$ *has a closed graph if and only if it is closed.*

Upper Semi-Continuity + Closed-Valuedness

Proposition 3.4

If F is upper semi-continuous and closed-valued, then it has a closed graph.

Proof

- Let $y^m \in F(x^m)$ for all $m \in \mathbb{N}$ and $(x^m, y^m) \to (\bar{x}, \bar{y}) \in X \times Y$.
- Take any $\varepsilon > 0$.
- $B_\varepsilon(F(\bar{x}))$ being an open neighborhood of $F(\bar{x})$, there exists an open neighborhood U of $\bar{x}$ such that $F(U) \subset B_\varepsilon(F(\bar{x}))$ by the upper semi-continuity of F at $\bar{x}$.
- Since $x^m \to \bar{x}$, there exists M such that for all $m \geq M$, $x^m \in U$ and hence $y^m \in F(U) \subset B_\varepsilon(F(\bar{x}))$.

  Therefore, we have $\bar{y} \in \bar{B}_\varepsilon(F(\bar{x}))$.
- Since $\varepsilon > 0$ has been taken arbitrarily and since $F(\bar{x})$ is closed, we have $\bar{y} \in F(\bar{x})$ (by Proposition 2.9).

Upper Semi-Continuity + Compact-Valuedness

Proposition 3.5

For correspondences $F\colon X \to Y$ and $G\colon X \to Y$, define the correspondence $F \cap G\colon X \to Y$ by $(F \cap G)(x) = F(x) \cap G(x)$ for all $x \in X$.
If

1. *F has a closed graph, and*
2. *G is upper semi-continuous and compact-valued,*

then $F \cap G$ is upper semi-continuous and compact-valued.

Proof

- Take any $\bar{x} \in X$, and consider any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x}$.

  Let $\{y^m\}$ be any sequence such that $y^m \in (F \cap G)(x^m) = F(x^m) \cap G(x^m)$ for all m.

- Since $y^m \in G(x^m)$ for all m, and by the upper semi-continuity of G at $\bar{x}$ and the compactness of $G(\bar{x})$, there exist a subsequence $\{y^{m(k)}\}$ and $\bar{y} \in G(\bar{x})$ such that $y^{m(k)} \to \bar{y}$.

- Since $y^m \in F(x^m)$ for all m, we thus have a sequence $\{(x^{m(k)}, y^{m(k)})\} \subset \mathrm{Graph}(F)$ that converges to $(\bar{x}, \bar{y})$.

  By the closedness of $\mathrm{Graph}(F)$, we have $(\bar{x}, \bar{y}) \in \mathrm{Graph}(F)$, i.e., $\bar{y} \in F(\bar{x})$.

- Hence, we have $\bar{y} \in (F \cap G)(\bar{x})$.

  The conclusion therefore follows from Proposition 3.1.

Upper Semi-Continuity + Compact-Valuedness

Proposition 3.6

For a correspondence $F\colon X \to Y$, consider the following conditions:

1. *F is upper semi-continuous and compact-valued.*
2. *F has a closed graph and the images of compact sets are compact.*
3. *F has a closed graph and the images of compact sets are bounded.*
4. *F has a closed graph and is locally bounded.*

We have the following:

- *$1 \Leftrightarrow 2 \Rightarrow 3 \Leftrightarrow 4$.*
- *If Y is closed, $3 \Rightarrow 2$ (hence these conditions are equivalent).*

- Thus, if Y is closed, then our definition is equivalent to that in MWG (condition 3) for compact-valued correspondences.

Proof

- $1 \Rightarrow 2$:

  By Propositions 3.2 and 3.4.

- $2 \Rightarrow 1$:

  Take any sequence $\{x^m\} \subset X$ such that $x^m \to \bar{x} \in X$ and any sequence $\{y^m\} \subset Y$ such that $y^m \in F(x^m)$ for all $m \in \mathbb{N}$.

  Since $A = \{x^m \mid m \in \mathbb{N}\} \cup \{\bar{x}\}$ is compact, $\{y^m\} \subset F(A)$ has a convergent subsequence with a limit $\bar{y} \in F(A)$ by the compactness of $F(A)$, where $\bar{y} \in F(\bar{x})$ by the closedness of the graph.

  Therefore, the conclusion follows by Proposition 3.1.

- $2 \Rightarrow 3$:

  Immediate.

Proof

- $3 \Rightarrow 4$:

  Suppose that F is not locally bounded,
  i.e., there exists some $\bar{x} \in X$ such that $F(B_\varepsilon(\bar{x}) \cap X)$ is not bounded for every $\varepsilon > 0$.

  For each $m \in \mathbb{N}$, let $y^m \in F(B_{1/m}(\bar{x}) \cap X)$ be such that $\|y^m\| > m$, and let $x^m \in B_{1/m}(\bar{x}) \cap X$ be such that $y^m \in F(x^m)$.

  By construction, $x^m \to \bar{x}$.

  Thus we have found a compact set $\{x^m \mid m \in \mathbb{N}\} \cup \{\bar{x}\}$ whose image is not bounded.

Proof

- $4 \Rightarrow 3$:

  Suppose that there exists a compact set $A \subset X$ such that $F(A)$ is not bounded.

  For each $m \in \mathbb{N}$, let $y^m \in F(A)$ be such that $\|y^m\| > m$, and let $x^m \in A$ be such that $y^m \in F(x^m)$.

  By the compactness of A, $\{x^m\}$ has a convergent subsequence $\{x^{m(k)}\}$ with a limit $\bar{x} \in A$.

  For any $\varepsilon > 0$, $F(B_\varepsilon(\bar{x}) \cap X)$ contains $\{y^{m(k)}\}_{k \geq K}$ for some K, which is unbounded.

Proof

- $3 \Rightarrow 2$ under the closedness of Y:

  Let $A \subset X$ be a compact set.

  Take any $\{y^m\} \subset F(A)$, and let $\{x^m\} \subset A$ be such that $y^m \in F(x^m)$ for all $m \in \mathbb{N}$.

  By the compactness of A and the boundedness of $F(A)$, $\{(x^m, y^m)\}$ has a convergent subsequence $\{(x^{m(k)}, y^{m(k)})\}$ with a limit $(\bar{x}, \bar{y}) \in A \times \mathbb{R}^K$.

  By the closedness of Y, $\bar{y} \in Y$, and therefore, by the closedness of the graph of F, $\bar{y} \in F(\bar{x}) \subset F(A)$.

  This implies that $F(A)$ is compact.

Upper Semi-Continuity + Compact-Valuedness

Corollary 3.7

Suppose that Y is compact.
$F\colon X \to Y$ is upper semi-continuous and compact-valued
if and only if it has a closed graph.

- Thus, if Y is compact, then our definition is equivalent to that in Debreu for compact-valued correspondences.

Lower Semi-Continuity

Let X and Y be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively.

Definition 3.4

- A correspondence $F\colon X \to Y$ is lower semi-continuous at $\bar{x} \in X$ if
  for any open set V (relative to Y) such that $F(\bar{x}) \cap V \neq \emptyset$, there exists an open neighborhood U (relative to X) of $\bar{x}$ such that $F(z) \cap V \neq \emptyset$ for all $z \in U$.
- For $A \subset X$, $F\colon X \to Y$ is lower semi-continuous on A if it is lower semi-continuous at all $\bar{x} \in A$.
- $F\colon X \to Y$ is lower semi-continuous if it is lower semi-continuous on X.

Lower Semi-Continuity

Proposition 3.8

For a correspondence $F\colon X \to Y$, the following statements are equivalent:

1. *F is lower semi-continuous at $\bar{x}$.*
2. *For any sequence $\{x^m\} \subset X$ with $x^m \to \bar{x}$ and any $y \in F(\bar{x})$, there exist a subsequence $\{x^{m(k)}\}$ of $\{x^m\}$ and a sequence $\{y^k\} \subset Y$ such that $y^k \in F(x^{m(k)})$ for all $k \in \mathbb{N}$ and $y^k \to y$.*
3. *For any sequence $\{x^m\} \subset X$ with $x^m \to \bar{x}$ and any $y \in F(\bar{x})$, there exist a sequence $\{y^m\} \subset Y$ and $M \in \mathbb{N}$ such that $y^m \in F(x^m)$ for all $m \geq M$ and $y^m \to y$.*

Lower Semi-Continuity

- Thus, our definition is equivalent to that in MWG.
- If F is nonempty-valued, then the proposition holds with $M = 1$.

  Thus, our definition is equivalent to that in Debreu for nonempty-valued correspondences.

Continuity

Let X and Y be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively.

Definition 3.5

- A correspondence $F\colon X \to Y$ is continuous at $\bar{x} \in X$ if it is both upper and lower semi-continuous at $\bar{x}$.
- For $A \subset X$, $F\colon X \to Y$ is continuous on A if it is continuous at all $\bar{x} \in A$.
- $F\colon X \to Y$ is continuous if it is continuous on X.

Example

Let X and A be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively.

Given a function $f\colon X \times A \to \mathbb{R}$, define the correspondence $F\colon A \to X$ by $F(\alpha) = \{x \in X \mid f(x, \alpha) \geq 0\}$.

Proposition 3.9

If f is upper semi-continuous, then F has a closed graph.

Proposition 3.10

If

- *for each $x \in X$, $f(x, \alpha)$ is lower semi-continuous in α, and*
- *for each $\alpha \in A$, for any $x \in X$ such that $f(x, \alpha) \geq 0$, and for any $\varepsilon > 0$, there exists $x' \in B_\varepsilon(x) \cap X$ such that $f(x', \alpha) > 0$,*

then F is lower semi-continuous.

Example

- The correspondence $B\colon \mathbb{R}^N_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N_+$ defined by

$$B(p,w) = \{x \in \mathbb{R}^N_+ \mid p \cdot x \leq w\}.$$

is continuous.

Example

- For a function $u\colon \mathbb{R}_+^N \to \mathbb{R}$, define the correspondence $V\colon \mathbb{R} \to \mathbb{R}_+^N$ by

$$V(t) = \{x \in \mathbb{R}_+^N \mid u(x) \geq t\}.$$

  - If u is upper semi-continuous, then V has a closed graph (but may not be upper semi-continuous in general).
  - If u is locally insatiable, then V is lower semi-continuous.

Singleton Values

For a correspondence $F\colon X \to Y$ and a function $f\colon X \to Y$, f is a *selection* of F if $f(x) \in F(x)$ for all $x \in X$.

Proposition 3.11

For a correspondence $F\colon X \to Y$, suppose that $F(\bar{x})$ is a singleton set.

- *If F is upper semi-continuous at $\bar{x}$, then any selection of F is continuous at $\bar{x}$.*
- *If there exists a selection continuous at $\bar{x}$, then F is lower semi-continuous at $\bar{x}$.*

Singleton Values

Proposition 3.12

For a function $f\colon X \to Y$, define the correspondence $F\colon X \to Y$ by $F(x) = \{f(x)\}$ for all $x \in X$.

- *If f is continuous, then F is upper semi-continuous.*
- *If F is lower semi-continuous, then f is continuous.*

Parametric Constrained Optimization

Let X and A be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively.

- For a function $f\colon X \times A \to \mathbb{R}$ and a nonempty-valued correspondence $\Gamma\colon A \to X$, consider the maximization problem

$$\max_x f(x, \alpha) \qquad \text{s.t. } x \in \Gamma(\alpha).$$

- If f is continuous and Γ is compact-valued, then by the Extreme Value Theorem, a solution exists $\forall\, \alpha \in A$.
- I.e., the value function $v(\alpha) = \max\limits_{x \in \Gamma(\alpha)} f(x, \alpha)$ is well defined, and the argmax correspondence $X^*(\alpha) = \arg\max\limits_{x \in \Gamma(\alpha)} f(x, \alpha)$ is nonempty-valued (and in fact also compact-valued).
- What are the continuity properties of v and X^*?

Theorem of the Maximum

Let X and A be nonempty subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively.

For a function $f\colon X \times A \to \mathbb{R}$ and a correspondence $\Gamma\colon A \to X$, define the function $v\colon A \to [-\infty, \infty]$ by

$$v(\alpha) = \sup_{x \in \Gamma(\alpha)} f(x, \alpha)$$

(let $v(\alpha) = -\infty$ if $\Gamma(\alpha) = \emptyset$).

Proposition 3.13

If Γ is lower semi-continuous and f is lower semi-continuous, then v is lower semi-continuous.

Proposition 3.14

If Γ is nonempty- and compact-valued and upper semi-continuous and f is upper semi-continuous, then v is upper semi-continuous.

Proof of Proposition 3.13

- Fix any $c \in \mathbb{R}$.
  We want to show that $\{\alpha \in A \mid v(\alpha) \leq c\}$ is closed.
- Suppose that $v(\alpha^m) \leq c$ and $\alpha^m \to \bar{\alpha} \in A$.
  We want to show that $f(x, \bar{\alpha}) \leq c$ for all $x \in \Gamma(\bar{\alpha})$.
- Take any $x \in \Gamma(\bar{\alpha})$.

  By the lower semi-continuity of Γ at $\bar{\alpha}$, we have a sequence $\{x^m\} \subset X$ such that $x^m \in \Gamma(\alpha^m)$ (for large m) and $x^m \to x$.
- Then $f(x^m, \alpha^m) \leq v(\alpha^m) \leq c$,
  but by the lower semi-continuity of f, we have $f(x, \bar{\alpha}) \leq c$.

Proof of Proposition 3.14

- Fix any $c \in \mathbb{R}$.
  We want to show that $\{\alpha \in A \mid v(\alpha) \geq c\}$ is closed.
- Suppose that $v(\alpha^m) \geq c$ and $\alpha^m \to \bar{\alpha} \in A$.
  We want to show that $f(x, \bar{\alpha}) \geq c$ for some $x \in \Gamma(\bar{\alpha})$.
- For each m, by the nonemptiness and compactness of $\Gamma(\alpha^m)$ and the upper semi-continuity of $f(x, \alpha^m)$ in x,
  we can take an $x^m \in \Gamma(\alpha^m)$ such that
  $f(x^m, \alpha^m) = v(\alpha^m) \geq c$.
- By the upper semi-continuity of Γ at $\bar{\alpha}$ and the compactness of $\Gamma(\bar{\alpha})$, there exist a subsequence $\{x^{m(k)}\}$ of $\{x^m\}$ and $\bar{x} \in \Gamma(\bar{\alpha})$ such that $x^{m(k)} \to \bar{x}$.
- By the upper semi-continuity of f, we have $f(\bar{x}, \bar{\alpha}) \geq c$.

Theorem of the Maximum

Define the correspondence $X^*\colon A \to X$ by

$$X^*(\alpha) = \{x \in X \mid x \in \Gamma(\alpha) \text{ and } f(x, \alpha) = v(\alpha)\}.$$

Proposition 3.15

Suppose that

- Γ *is nonempty- and compact-valued and continuous, and*
- f *is continuous.*

Then

1. X^* *is nonempty- and compact-valued,*
2. v *is continuous, and*
3. X^* *is upper semi-continuous.*

Proof of Proposition 3.15

1. By the Extreme Value Theorem.
2. By Propositions 3.13 and 3.14.
3. The correspondence $\hat{X}(\alpha) = \{x \in X \mid f(x, \alpha) = v(\alpha)\}$ has a closed graph by the continuity of f and v.

   Therefore, X^* $(= \hat{X} \cap \Gamma)$ is upper semi-continuous by Proposition 3.5.

Utility Maximization

For $p \in \mathbb{R}^N_{++}$ and $w \in \mathbb{R}_{++}$,

$$\begin{aligned} \max_{x \in \mathbb{R}^N_+} \quad & u(x) \\ \text{s. t.} \quad & p \cdot x \leq w. \end{aligned}$$

- Indirect utility function $\cdots$ optimal value function:
  the function $v \colon \mathbb{R}^N_{++} \times \mathbb{R}_{++} \to (-\infty, \infty]$ defined by
  $$v(p, w) = \sup\{u(x) \mid x \in B(p, w)\}.$$
- Walrasian demand correspondence $\cdots$ optimal solution correspondence:
  the correspondence $x \colon \mathbb{R}^N_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N_+$ defined by
  $$\begin{aligned} x(p, w) = \{x^* \in \mathbb{R}^N_+ \mid x^* \in B(p, w) \text{ and} \\ u(x^*) \geq u(x) \text{ for all } x \in B(p, w)\}. \end{aligned}$$

Proposition 3.16

Assume that u is continuous.
Then v is continuous, and x is nonempty- and compact-valued and upper semi-continuous.

Proof

Since B is nonempty- and compact-valued and continuous, the claim follows from the Theorem of the Maximum.

Expenditure Minimization

Write $\bar{v} = \sup u(\mathbb{R}^N_+)$, and assume $u(0) < \bar{v}$.

For $p \in \mathbb{R}^N_{++}$ and $t \in [u(0), \bar{v})$,

$$\begin{aligned} \min_{x \in \mathbb{R}^N_+} \quad & p \cdot x \\ \text{s. t.} \quad & u(x) \geq t. \end{aligned}$$

- Expenditure function $\cdots$ optimal value function:
  the function $e \colon \mathbb{R}^N_{++} \times [u(0), \bar{v}) \to \mathbb{R}$ defined by
  $$e(p, t) = \inf\{p \cdot x \mid u(x) \geq t\}.$$
- Hicksian demand correspondence $\cdots$ optimal solution correspondence:
  the correspondence $h \colon \mathbb{R}^N_{++} \times [u(0), \bar{v}) \to \mathbb{R}^N_+$ defined by
  $$\begin{aligned} h(p, t) = \{x^* \in \mathbb{R}^N_+ \mid u(x^*) \geq t \text{ and } \\ p \cdot x^* \leq p \cdot x \text{ for all } x \in \mathbb{R}^N_+ \text{ such that } u(x) \geq t\}. \end{aligned}$$

Proposition 3.17

Assume that u is upper semi-continuous.

1. $p \mapsto e(p,t)$ *is continuous, and* $p \mapsto h(p,t)$ *is nonempty- and compact-valued and upper semi-continuous.*
2. e *is lower semi-continuous.*
3. *If u is locally insatiable, then e is continuous, and h is nonempty- and compact-valued and upper semi-continuous.*

Proof

- The objective function $p \cdot x$ is continuous in (x, p).
- Let $V(t) = \{x \in \mathbb{R}^N_+ \mid u(x) \geq t\}$ (not bounded in general).

2, 3.

- Fix any $(\bar{p}, \bar{t}) \in \mathbb{R}^N_{++} \times [u(0), \bar{v})$.
- Since $\bar{t} < \bar{v} = \sup u(\mathbb{R}^N_+)$, there exists some $x^0 \in \mathbb{R}^N_+$ such that $u(x^0) > \bar{t}$.
- Let $U^0 = [u(0), u(x^0))$ $(\neq \emptyset)$.
- For $t \in U^0$, define
  $V^0(t) = V(t) \cap \{x \in \mathbb{R}^N_+ \mid \bar{p} \cdot x \leq \bar{p} \cdot x^0 + 1\}$.
- For all $t \in U^0$, $V^0(t) \neq \emptyset$ since $x^0 \in V^0(t)$.

Proof

- Since $\{x \in \mathbb{R}^N_+ \mid \bar{p} \cdot x \leq \bar{p} \cdot x^0 + 1\}$ is a neighborhood of $\{x \in \mathbb{R}^N_+ \mid \bar{p} \cdot x \leq \bar{p} \cdot x^0\}$ and $p \mapsto \{x \in \mathbb{R}^N_+ \mid p \cdot x \leq p \cdot x^0\}$ is upper semi-continuous,

  we can take an open neighborhood $P^0 \subset \mathbb{R}^N_{++}$ of $\bar{p}$ such that $\{x \in \mathbb{R}^N_+ \mid p \cdot x \leq p \cdot x^0\} \subset \{x \in \mathbb{R}^N_+ \mid \bar{p} \cdot x \leq \bar{p} \cdot x^0 + 1\}$ for all $p \in P^0$.
- By construction, for all $(p, t) \in P^0 \times U^0$, $-e(p, t) = \sup\{-(p \cdot x) \mid x \in V^0(t)\}$ and $h(p, t) = \{x \in \mathbb{R}^N_+ \mid x \in V^0(t) \text{ and } p \cdot x = e(p, t)\}$.
- $(p, t) \mapsto V^0(t)$ has a closed graph by the upper semi-continuity of u, and $V^0(t)$ is contained in the compact set $\{x \in \mathbb{R}^N_+ \mid \bar{p} \cdot x \leq \bar{p} \cdot x^0 + 1\}$ for all $(p, t) \in P^0 \times U^0$.

  Therefore, by Corollary 3.7, $(p, t) \mapsto V^0(t)$ is upper semi-continuous and compact-valued.
- Thus by Proposition 3.14, $-e$ is upper semi-continuous.

Proof

- $(p, t) \mapsto V(t)$ is lower semi-continuous if u is locally insatiable.

  Thus by Proposition 3.13, $-e$ is lower semi-continuous.
- The upper semi-continuity of h follows as in the proof of the Theorem of the Maximum.

Proof

1.

- With fixed $\bar{t}$, $p \mapsto V^0(\bar{t})$ is continuous (and compact-valued).
- Thus by the Theorem of the Maximum, $p \mapsto h(p, \bar{t})$ is nonempty- and compact-valued and upper semi-continuous on P^0, and $p \mapsto -e(p, \bar{t})$ is continuous on P^0.

Profit Maximization

For $Y \subset \mathbb{R}^N$ with $Y \neq \emptyset$ and for $p \in \mathbb{R}^N_{++}$,

$$\begin{aligned} \max_{y \in \mathbb{R}^N} \quad & p \cdot y \\ \text{s. t.} \quad & y \in Y. \end{aligned}$$

- Profit function $\cdots$ optimal value function:
  the function $\pi \colon \mathbb{R}^N_{++} \to (-\infty, \infty]$ defined by

  $$\pi(p) = \sup\{p \cdot y \mid y \in Y\}.$$

- Supply correspondence $\cdots$ optimal solution correspondence:
  the correspondence $S \colon \mathbb{R}^N_{++} \to \mathbb{R}^N$ defined by

  $$S(p) = \{y^* \in \mathbb{R}^N \mid y^* \in Y \text{ and } p \cdot y^* \geq p \cdot y \text{ for all } y \in Y\}.$$

Proposition 3.18

Suppose that Y is nonempty, closed, and convex.
If $S(\bar{p})$ is nonempty and bounded, then there exists an open neighborhood $P^0 \subset \mathbb{R}^N_{++}$ of $\bar{p}$ such that

1. $S(p) \neq \emptyset$ *for all* $p \in P^0$ *and* $\bigcup_{p \in P^0} S(p)$ *is bounded,*
2. S *is upper semi-continuous on* P^0*, and*
3. π *is continuous on* P^0*.*

Proof

- By the closedness and convexity of $Y \neq \emptyset$, the continuity of $p \cdot y$ in (y, p), and the linearity of $p \cdot y$ in y,
  there exists an open neighborhood $P^0 \subset \mathbb{R}^N_{++}$ of $\bar{p}$ such that $S(p) \neq \emptyset$ for all $p \in P^0$ and $\bigcup_{p \in P^0} S(p)$ is bounded.

  (See Lemma A.4 in Oyama and Takenawa (2018).)
- For such P^0, let $Y^0 = \mathrm{Cl} \bigcup_{p \in P^0} S(p)$, which is nonempty and compact.
- Then, for $p \in P^0$, we have $\pi(p) = \max\{p \cdot y \mid y \in Y^0\}$ and $S(p) = \arg\max\{p \cdot y \mid y \in Y^0\}$.
- Therefore, by the compactness of Y^0 and the continuity of $p \cdot y$ in (y, p), the Theorem of the Maximum applies.