5. Differentiation I

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Differentiation in One Variable

Let $I \subset \mathbb{R}$ be a nonempty interval.

Definition 5.1

- A function $f\colon I \to \mathbb{R}$ is *differentiable* at $\bar{x} \in I$ if

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x})}{h}$$

exists, i.e., if there exists $a \in \mathbb{R}$ such that
for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |h| < \delta,\ \bar{x} + h \in I \Longrightarrow \left| \frac{f(\bar{x}+h) - f(\bar{x}) - ah}{h} \right| < \varepsilon.$$

- In this case, the limit a is called the *differential coefficient* of f at $\bar{x}$, and denoted by $f'(\bar{x})$ or $\frac{df}{dx}(\bar{x})$.

- For $I' \subset I$, f is differentiable on I' if f is differentiable at all $\bar{x} \in I'$.

- f is differentiable if f is differentiable on I.

- If f is differentiable on I',
  the function $x \mapsto f'(x)$ from I' to $\mathbb{R}$ is called
  the *derivative function* (or *derivative*) of f and
  denoted by f' or $\frac{df}{dx}$.

- If f is differentiable and f' is continuous, then f is said to be *continuously differentiable* or *of class C^1*.

Little o Notation

- If $\lim_{x \to \bar{x}} g(x) = 0$ and $\lim_{x \to \bar{x}} \frac{f(x)}{g(x)} = 0$, we write

  $$f(x) = o(g(x)) \text{ as } x \to \bar{x}.$$

- For example, $x^2 = o(x)$ as $x \to 0$.

  (I.e., x^2 is much smaller than x when $x \approx 0$.)
- By $f(x) = h(x) + o(g(x))$, we mean $f(x) - h(x) = o(g(x))$.
- If f is differentiable at $\bar{x}$, then

  $$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + o(x - \bar{x}) \text{ as } x \to \bar{x},$$

  or

  $$f(\bar{x} + \varepsilon) = f(\bar{x}) + f'(\bar{x})\varepsilon + o(\varepsilon) \text{ as } \varepsilon \to 0.$$

  (Often, "as $\varepsilon \to 0$" is omitted.)

Differentiability and Continuity

Proposition 5.1

If f is differentiable at $\bar{x}$, then it is continuous at $\bar{x}$.

Proof

$$\begin{aligned}\lim_{x \to \bar{x}} f(x) &= \lim_{x \to \bar{x}} \big(f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + o(x - \bar{x}) \big) \\ &= f(\bar{x}).\end{aligned}$$

- The converse does not hold.

  For example, the continuous function $x \mapsto |x|$ is not differentiable at 0.

First-Order Condition for Optimality

Proposition 5.2

Let $I \subset \mathbb{R}$ be a nonempty open interval.
For $f\colon I \to \mathbb{R}$ and $x^* \in I$, if

- $f(x^*) \geq f(x)$ *for all* $x \in I$ *and*
- f *is differentiable at* x^*,

then $f'(x^*) = 0$.

Proof

- For any sufficiently small $\varepsilon > 0$, we have $\frac{f(x^*+\varepsilon)-f(x^*)}{\varepsilon} \leq 0$.
- Therefore,

$$f'(x^*) = \lim_{\varepsilon \searrow 0} \frac{f(x^* + \varepsilon) - f(x^*)}{\varepsilon} \leq 0.$$

- Similarly, we have $\frac{f(x^*)-f(x^*-\varepsilon)}{\varepsilon} \geq 0$.
- Therefore,

$$f'(x^*) = \lim_{\varepsilon \searrow 0} \frac{f(x^*) - f(x^* - \varepsilon)}{\varepsilon} \geq 0.$$

Mean Value Theorem

Proposition 5.3

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b), where $a < b$. Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof

Consider $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$.
Note that $g(a) = g(b)$ $(= 0)$.
Since g is continuous on the compact set $[a, b]$,
it has a maximum y^* and a minimum y^{**}.
If $y^* = y^{**}$, then the assertion obviously holds.

If $y^* \neq y^{**}$,
then a maximizer x^* exists in (a, b) in which case $g'(x^*) = 0$,
or a minimizer x^{**} exists in (a, b) in which case $g'(x^{**}) = 0$.

Applications

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b), where $a < b$.

- If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is nondecreasing on $[a, b]$ (i.e., $f(x_1) \leq f(x_2)$ for any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$)

  (The converse also holds.)

- If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$ (i.e., $f(x_1) < f(x_2)$ for any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$).

- If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

- The following is *false*:
  "if f is strictly increasing on $[a, b]$, then $f'(x) > 0$ for all $x \in (a, b)$".
  Find a counter-example.

Proof

Take any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$.

By the Mean Value Theorem, there exists some $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Thus,

- $f'(x) \geq 0$ for all $x \in (a, b) \Rightarrow f(x_1) \leq f(x_2)$;
- $f'(x) > 0$ for all $x \in (a, b) \Rightarrow f(x_1) < f(x_2)$;
- $f'(x) = 0$ for all $x \in (a, b) \Rightarrow f(x_1) = f(x_2)$.

Inverse Function Theorem: One Variable Case

Proposition 5.4

Let $I \subset \mathbb{R}$ be a nonempty open interval.
Suppose that $f\colon I \to \mathbb{R}$ is of class C^1 and $f'(\bar{x}) \neq 0$ for $\bar{x} \in I$.
Then there exists an open interval $J \subset I$ containing $\bar{x}$ that satisfies the following:

- $f|_J\colon J \to f(J)$ *is a bijection;*
- $(f|_J)^{-1}\colon f(J) \to J$ *is of class C^1; and*
- $((f|_J)^{-1})'(y) = \frac{1}{f'((f|_J)^{-1}(y))}$ *for all $y \in f(J)$.*

- $f(J) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in J\}$.

Higher Order Derivatives

Let $I \subset \mathbb{R}$ be a nonempty interval.
Suppose that a function $f\colon I \to \mathbb{R}$ is differentiable on I.

- If the function f' is differentiable on I,
  then f is said to be twice differentiable, and
  the derivative function of f' is denoted by f'', or $\frac{d^2 f}{dx^2}$,
  and is called the 2nd derivative function of f.
- $\cdots$
- If the function $f^{(n-1)}$ is differentiable on I,
  then f is said to be n times differentiable, and
  the derivative function of $f^{(n-1)}$ is denoted by $f^{(n)}$, or $\frac{d^n f}{dx^n}$,
  and is called the nth derivative function of f,
  where $f^{(1)} = f'$.
- If f is n times differentiable and $f^{(n)}$ is continuous, then f is said to be n *times continuously differentiable* or *of class* C^n.

Taylor's Theorem: 2nd Order Case

Let $I \subset \mathbb{R}$ be a nonempty open interval.
Let $a, b \in I$ with $a < b$.

Proposition 5.5

1. *If $f\colon I \to \mathbb{R}$ is differentiable and f' is differentiable at a, then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + o((x-a)^2).$$

2. *If $f\colon I \to \mathbb{R}$ is twice differentiable, then there exists $c \in (a, b)$ such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(c)(b-a)^2.$$

Proof

2.

- Let $g(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}A(x-a)^2$, where A is a constant such that $g(b) = 0$, i.e.,

  $$A = 2\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$

  We want to show that $A = f''(c)$ for some $c \in (a,b)$.

- We have $g(a) = 0$, $g(b) = 0$, and $g'(a) = 0$.

- Since g is differentiable on I (and so on $[a,b]$), there is some $c_0 \in (a,b)$ such that $g'(c_0) = 0$ by the Mean Value Theorem.

- Since g' is differentiable on I (and so on $[a,b]$), there is some $c \in (a, c_0)$ such that $g''(c) = 0$ by the Mean Value Theorem.

- Since $g''(x) = f''(x) - A$, we have $A = f''(c)$.

Second-Order Sufficient Condition for Optimality

Proposition 5.6

Let $I \subset \mathbb{R}$ be a nonempty open interval.
For $f\colon I \to \mathbb{R}$ and $x^* \in I$, if

- f *is differentiable on* I *and* f' *is differentiable at* x^*,
- $f'(x^*) = 0$, *and*
- $f''(x^*) < 0$,

then x^* is a strict local maximizer of f, i.e., there exists $\delta > 0$ such that $f(x^*) > f(x)$ for all $x \in (x^* - \delta, x^* + \delta)$, $x \neq x^*$.

Proof

- Since $f'(x^*) = 0$, by Taylor's Theorem we have

$$f(x) = f(x^*) + \frac{1}{2} f''(x^*)(x - x^*)^2 + o((x - x^*)^2),$$

  i.e., $\displaystyle\lim_{x \to x^*} \frac{f(x) - f(x^*)}{(x - x^*)^2} = \frac{1}{2} f''(x^*)$.

- Since $f''(x^*) < 0$, we can take an $\varepsilon > 0$ such that $\frac{1}{2} f''(x^*) + \varepsilon < 0$.

- Given this $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in (x^* - \delta, x^* + \delta)$, $x \neq x^*$,

$$\frac{f(x) - f(x^*)}{(x - x^*)^2} < \frac{1}{2} f''(x^*) + \varepsilon < 0,$$

  so that $f(x) < f(x^*)$.

Second-Order Necessary Condition for Optimality

Proposition 5.7

Let $I \subset \mathbb{R}$ be a nonempty open interval.
For $f\colon I \to \mathbb{R}$ and $x^* \in I$, if

- *f is differentiable on I and f' is differentiable at x^*, and*
- *x^* is a maximizer of f,*

then $f''(x^*) \leq 0$.

Proof

By Proposition 5.6.