9. Envelope Theorem

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Parameterized Optimization (with Constant Constraints)

Let $X \subset \mathbb{R}^N$ be a nonempty set, $A \subset \mathbb{R}^S$ a nonempty open set.

For $f\colon X \times A \to \mathbb{R}$, consider the optimal value function

$$v(q) = \sup_{x \in X} f(x, q),$$

and the optimal solution correspondence

$$X^*(q) = \{x \in X \mid f(x, q) = v(q)\}.$$

We assume that $X^*(q) \neq \emptyset$ for all $q \in A$.

We want to investigate the marginal effects of changes in q on the value $v(q)$.

Formally, the *envelope theorem* gives

1. a sufficient condition under which v is differentiable, and
2. a formula for the derivative ("envelope formula").

Outline

- Envelope formula is best interpreted through FOC under the differentiability of the solution function (or selection) and the differentiability of f in (x, q),

  while these assumptions are irrelevant for the differentiability of v and deriving the formula.

- If we directly assume the differentiability of v, deriving the envelope formula is just a straightforward routine.

  Differentiability of v is the real content of envelope theorem.

- Non-differentiability of v is a typical case when there are more than one solutions.

- Provide a sufficient condition under which uniqueness of solution implies differentiability of v,

  with applications for the differentiability of support function (or profit function), indirect utility function, and expenditure function.

Main Reference

- D. Oyama and T. Takenawa, "On the (Non-)Differentiability of the Optimal Value Function When the Optimal Solution Is Unique," *Journal of Mathematical Economics* 76, 21-32 (2018).

Envelope Theorem via FOC

Proposition 9.1

Let $x(\cdot)$ be a selection of X^*,
i.e., a function such that $x(q) \in X^*(q)$ for all $q \in A$.
Assume that

1. *f is differentiable on $\operatorname{Int} X \times A$, and*
2. *$x(\bar{q}) \in \operatorname{Int} X$, and $x(\cdot)$ is differentiable at $\bar{q}$.*

Then, v is differentiable at $\bar{q}$, and

$$\frac{\partial v}{\partial q_s}(\bar{q}) = \frac{\partial f}{\partial q_s}(x(\bar{q}), \bar{q}), \qquad s = 1, \dots, S.$$

Proof

- By the assumptions, $v(q) = f(x(q), q)$ is differentiable at $\bar{q}$.
- We have

$$\begin{aligned}
\frac{\partial v}{\partial q_s}(\bar{q}) &= \frac{\partial}{\partial q_s} f(x(q), q)\Big|_{q=\bar{q}} \\
&= \sum_n \underbrace{\frac{\partial f}{\partial x_n}(x(\bar{q}), \bar{q})}_{= 0 \text{ by FOC}} \frac{\partial x_n}{\partial q_s}(\bar{q}) + \frac{\partial f}{\partial q_s}(x(\bar{q}), \bar{q}) \\
&= \frac{\partial f}{\partial q_s}(x(\bar{q}), \bar{q}).
\end{aligned}$$

- The change in the solution caused by the change in q has no first-order effect on the value;
- the only effect is the direct effect.

A Sufficient Condition for the Differentiability of $x(\cdot)$

Proposition 9.2

Assume that

1. X *is compact and* f *is continuous,*
2. *for each* $q \in A$, $X^*(q) = \{x(q)\} \subset \operatorname{Int} X$,
3. $\nabla_x f$ *exists and is continuously differentiable on* $\operatorname{Int} X \times A$, *and*
4. $|D_x^2 f(x(\bar{q}), \bar{q})| \neq 0$.

Then, $x(\cdot)$ *is continuously differentiable on a neighborhood of* $\bar{q}$.

Proof

- By assumptions, $x(\cdot)$ is continuous by the Theorem of Maximum.
- By the FOC, $\nabla_x f(x(q), q) = 0$ for all $q \in A$.
- By assumptions, $\nabla_x f(x, q) = 0$ is uniquely solved locally as $x = \eta(q)$ and η is continuously differentiable by the Implicit Function Theorem.

  I.e., there exist open neighborhoods U and V of $x(\bar{q})$ and $\bar{q}$, respectively, such that $\nabla_x f(x, q) = 0$ if and only if $x = \eta(q)$.
- By the continuity of $x(\cdot)$, there exists an open neighborhood $V' \subset V$ of $\bar{q}$ such that $x(q) \in U$ for all $q \in V'$.
- By the FOC $\nabla_x f(x(q), q) = 0$, it follows that $x(q) = \eta(q)$ for all $q \in V'$.

Envelope Formula

If we directly assume the differentiability of the value function v, neither the differentiability of $f(x, q)$ in x nor that of $x(q)$ in q is needed in deriving the envelope formula.

Proposition 9.3

Assume that

1. *for all $x \in X$, $f(x, \cdot)$ is differentiable at $\bar{q}$, and*
2. *v is differentiable at $\bar{q}$.*

Then, for any $\bar{x} \in X^*(\bar{q})$,

$$\frac{\partial v}{\partial q_s}(\bar{q}) = \frac{\partial f}{\partial q_s}(\bar{x}, \bar{q}), \qquad s = 1, \ldots, S.$$

Proof

- Fix any $\bar{x} \in X^*(\bar{q})$.
- Define the function

$$g(q) = f(\bar{x}, q) - v(q).$$

  By assumption, g is differentiable at $\bar{q}$.
- By definition,
  - $g(q) \leq 0$ for all $q \in A$, and
  - $g(\bar{q}) = 0$.
- Thus, g is maximized at $\bar{q}$, so that by FOC we have

$$0 = \frac{\partial g}{\partial q_s}(\bar{q}) = \frac{\partial f}{\partial q_s}(\bar{x}, \bar{q}) - \frac{\partial v}{\partial q_s}(\bar{q}).$$

Example: Non-Differentiable Value Function

Consider

$$f(x,q) = -\frac{1}{4}x^4 - \frac{q}{3}x^3 + \frac{1}{2}x^2 + qx - \frac{1}{4}, \quad q \in [-1,1],$$

where $f_x(x,q) = -(x+1)(x+q)(x-1)$.

Then we have

$$v(q) = \frac{2}{3}|q|, \quad X^*(q) = \begin{cases} \{-1\} & \text{if } q < 0, \\ \{-1,1\} & \text{if } q = 0, \\ \{1\} & \text{if } q > 0. \end{cases}$$

- At $q = 0$, v is not differentiable, and
- there are two optimal solutions.

A Sufficient Condition for Differentiability of v

Proposition 9.4

Assume that

1. X^* *has a selection* $x(\cdot)$ *continuous at* $\bar{q}$*, and*
2. *for all* $x \in X$*,* $f(x, \cdot)$ *is differentiable, and* $\nabla_q f$ *is continuous in* (x, q)*.*

Then v *is differentiable at* $\bar{q}$ *with*

$$\frac{\partial v}{\partial q_s}(\bar{q}) = \frac{\partial f}{\partial q_s}(x(\bar{q}), \bar{q}), \qquad s = 1, \ldots, S.$$

Proof

See: Oyama and Takenawa, Proposition A.1.

Corollary 9.5

Assume that

1. X^* *is upper semi-continuous with* $X^*(q) \neq \emptyset$ *for all* $q \in A$,
2. $X^*(\bar{q}) = \{\bar{x}\}$, *and*
3. *for all* $x \in X$, $f(x, \cdot)$ *is differentiable, and* $\nabla_q f$ *is continuous in* (x, q).

Then v *is differentiable at* $\bar{q}$ *with*

$$\frac{\partial v}{\partial q_s}(\bar{q}) = \frac{\partial f}{\partial q_s}(\bar{x}, \bar{q}), \qquad s = 1, \ldots, S.$$

Proof

- Assumptions 1 and 2 imply that any selection of X^* is continuous at $\bar{q}$.
- Thus the conclusion follows from Proposition 9.4.

Remark

- A sufficient condition for Assumption 1 is that X is compact and f is continuous, due to the Theorem of the Maximum.

Example: Non-Differentiable Value Function

- Even if an optimal solution is unique, the value function may not be differentiable.
- In fact, there exists a continuous function $f \colon X \times A \to \mathbb{R}$ such that

  1. $X^*(q)$ is a singleton for all q and is continuous in q (as a single-valued function), and
  2. f is differentiable in q,

  but v is not differentiable at some q.

See: Oyama and Takenawa, Example 2.1.

Differentiability of the Support Function

For $K \subset \mathbb{R}^N$, $K \neq \emptyset$, and $p \in \mathbb{R}^N$, consider the support function of K,

$$\pi_K(p) = \sup_{x \in K} p \cdot x,$$

and the optimal solution correspondence,

$$S_K(p) = \{x \in \mathbb{R}^N \mid x \in K,\ \pi_K(p) = p \cdot x\}.$$

If K is the production set of a firm, π_K is the profit function and S_K is the supply correspondence (defined for all $p \in \mathbb{R}^N$).

- If K is closed and convex and if $S_K(\bar{p})$ is nonempty and bounded, then there exists an open neighborhood P^0 of $\bar{p}$ such that S_K is nonempty-valued and upper semi-continuous on P^0. (Proposition 3.18)

Proposition 9.6

Let $K \subset \mathbb{R}^N$ be a nonempty closed convex set, and $\pi_K \colon \mathbb{R}^N \to (-\infty, \infty]$ its support function, i.e., $\pi_K(p) = \sup_{x \in K} p \cdot x$.
Let $\bar{p} \in \mathbb{R}^N$ be such that $\pi_K(\bar{p}) < \infty$.
Then π_K is differentiable at $\bar{p}$ if and only if there is a unique $\bar{x} \in K$ such that $\pi_K(\bar{p}) = \bar{p} \cdot \bar{x}$.
In this case, $\nabla \pi_K(\bar{p}) = \bar{x}$.

Proof

“If” part

- By the closedness and convexity of $K \neq \emptyset$, it follows from Proposition 3.18 that there exists an open neighborhood P^0 of $\bar{p}$ such that S_K is nonempty-valued and upper semi-continuous on P^0.
- The function $f(x, p) = p \cdot x$ is differentiable in p, and $\nabla_p f(x, p) = x$ is continuous in (x, p).
- With $S_K(\bar{p}) = \{\bar{x}\}$, the conclusion follows from Corollary 9.5.

Proof

"Only if" part

- By the definition, $S_K(p) \subset \partial \pi_K(p)$.
- Since π_K is convex, the differentiability of π_K at $\bar{p}$ implies $\partial \pi_K(\bar{p}) = \{\nabla \pi_K(\bar{p})\}$.
- The nonemptiness of $S_K(\bar{p})$ follows from the differentiability of π_K by an elementary argument under the closedness of K (see Oyama and Takenawa, Lemma A.5).

  Hence, $S_K(\bar{p})$ is a singleton.
- By the differentiability of π_K, we have $\nabla \pi_K(\bar{p}) = \nabla_p (p \cdot x)|_{p=\bar{p}, x=\bar{x}} = \bar{x}$ for all $\bar{x} \in S_K(\bar{p})$.

The convexity of K can be dropped if K is compact, in which case $\mathrm{Co}\, K$ is closed.

Corollary 9.7

Let $K \subset \mathbb{R}^N$ be a nonempty compact set. Then π_K is differentiable at $\bar{p}$ if and only if there is a unique $\bar{x} \in K$ such that $\pi_K(\bar{p}) = \bar{p} \cdot \bar{x}$. In this case, $\nabla \pi_K(\bar{p}) = \bar{x}$.

Proof

- Show that $S_{\mathrm{Co}\,K} = \mathrm{Co}\,S_K$.
- $\mathrm{Co}\,K$ is nonempty and closed if K is nonempty and compact.
- Therefore, it follows from Proposition 9.6 that

$$\begin{aligned}
&\pi_K = \pi_{\mathrm{Co}\,K} \text{ is differentiable at } \bar{p} \\
&\iff S_{\mathrm{Co}\,K}(\bar{p}) = \mathrm{Co}\,S_K(\bar{p}) \text{ is a singleton} \\
&\iff S_K(\bar{p}) \text{ is a singleton,}
\end{aligned}$$

in which case $S_K(\bar{p}) = \{\nabla \pi_K(\bar{p})\}$.

Differentiability of the Indirect Utility Function

For $p \in \mathbb{R}^N_{++}$ and $w \in \mathbb{R}_{++}$, consider the indirect utility function,

$$v(p, w) = \sup\{u(x) \mid x \in B(p, w)\},$$

and the Walrasian demand correspondence,

$$x(p, w) = \{x \in \mathbb{R}^N_+ \mid x \in B(p, w),\ u(x) = v(p, w)\},$$

where $B(p, w) = \{x \in \mathbb{R}^N_+ \mid p \cdot x \leq w\}$.

- If u is continuous, then x is nonempty- and compact-valued and upper semi-continuous.

  (Proposition 3.16)

Proposition 9.8

Assume that

1. u *is locally insatiable and continuous,*
2. $x(\bar{p}, \bar{w}) = \{\bar{x}\}$, *and*
3. *for some* j *with* $\bar{x}_j > 0$ *and for some neighborhoods* X_j^0 *and* X_{-j}^0 *of* $\bar{x}_j$ *and* $\bar{x}_{-j}$ *in* $\mathbb{R}_+$ *and* $\mathbb{R}_+^{N-1}$, *respectively,* $\frac{\partial u}{\partial x_j}$ *exists on* $X_j^0 \times X_{-j}^0$ *and is continuous in* x *at* $\bar{x}$.

Then v *is differentiable at* $(\bar{p}, \bar{w})$ *with*

$$\frac{\partial v}{\partial p_i}(\bar{p}, \bar{w}) = -\frac{\frac{\partial u}{\partial x_j}(\bar{x})}{\bar{p}_j}\bar{x}_i, \quad \frac{\partial v}{\partial w}(\bar{p}, \bar{w}) = \frac{\frac{\partial u}{\partial x_j}(\bar{x})}{\bar{p}_j}$$

for any j *satisfying the condition in 3.*

Proof (1/3)

- By the local insatiability, the inequality constraint $p \cdot x \leq w$ can be replaced by the equality constraint $p \cdot x = w$.
- Let $x(\bar{p}, \bar{w}) = \{\bar{x}\}$, where $\bar{p} \cdot \bar{x} = \bar{w}$.
- Let j, X_j^0, and X_{-j}^0 be as in Assumption 3, where $\bar{x}_j = \frac{1}{\bar{p}_j}\left(\bar{w} - \sum_{i \neq j} \bar{p}_i \bar{x}_i\right) \in X_j^0$.
- Write $x_{-j} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$, and let

$$f(x_{-j}, p, w) = u\left(\frac{1}{p_j}\left(w - \sum_{i \neq j} p_i x_i\right), x_{-j}\right).$$

- As long as $\frac{1}{p_j}\left(w - \sum_{i \neq j} p_i x_i\right) \in X_j^0$, f is well defined and continuous, and $\nabla_{(p,w)} f$ exists on a neighborhood of $(\bar{x}_{-j}, \bar{p}, \bar{w})$ and is continuous in (x_{-j}, p, w) at $(\bar{x}_{-j}, \bar{p}, \bar{w})$ by Assumption 3.

Proof (2/3)

- We claim that there exist open neighborhoods P^1 and W^1 of $\bar{p}$ and $\bar{w}$ and a compact neighborhood $X^1_{-j} \subset \mathbb{R}^{N-1}_+$ of $\bar{x}_{-j}$ such that

$$v(p, w) = \max_{x_{-j} \in X^1_{-j}} f(x_{-j}, p, w) \text{ for all } (p, w) \in P^1 \times W^1,$$

where

$$\arg\max_{x_{-j} \in X_{-j}} f(x_{-j}, \bar{p}, \bar{w}) = \{\bar{x}_{-j}\}.$$

- Then by Corollary 9.5, v is differentiable at $(\bar{p}, \bar{w})$, and

$$\frac{\partial v}{\partial p_i}(\bar{p}, \bar{w}) = \frac{\partial f}{\partial p_i}(\bar{x}_{-j}, \bar{p}, \bar{w}) = \frac{\partial u}{\partial x_j}(\bar{x}) \frac{1}{p_j}(-\bar{x}_i),$$
$$\frac{\partial v}{\partial w}(\bar{p}, \bar{w}) = \frac{\partial f}{\partial w}(\bar{x}_{-j}, \bar{p}, \bar{w}) = \frac{\partial u}{\partial x_j}(\bar{x}) \frac{1}{p_j}.$$

Proof (3/3)

- X^1_{-j}, P^1, and W^1 are constructed as follows:
- Since $\bar{x}_j = \frac{1}{\bar{p}_j}\left(\bar{w} - \sum_{i \neq j} \bar{p}_i \bar{x}_i\right) \in X^0_j$ and $\frac{1}{p_j}\left(w - \sum_{i \neq j} p_i x_i\right)$ is continuous in (x_{-j}, p, w), there exist open neighborhoods P^0 and W^0 of $\bar{p}$ and $\bar{w}$ and a compact neighborhood $X^1_{-j} \subset \mathbb{R}^{N-1}_+$ of $\bar{x}_{-j}$ such that $\frac{1}{p_j}\left(w - \sum_{i \neq j} p_i x_i\right) \in X^0_j$ for all $(x_{-j}, p, w) \in X^1_{-j} \times P^0 \times W^0$.
- Since $x(p, w)$ is upper semi-continuous and $x_{-j}(\bar{p}, \bar{w}) \subset X^1_{-j}$, we can take open neighborhoods $P^1 \subset P^0$ and $W^1 \subset W^0$ of $\bar{p}$ and $\bar{w}$ such that $x_{-j}(p, w) \subset X^1_{-j}$ for all $(p, w) \in P^1 \times W^1$.

Differentiability of the Expenditure Function

For $p \in \mathbb{R}^N_{++}$ and $t \in [u(0), \bar{u})$, where $\bar{u} = \sup_{x \in \mathbb{R}^N_+} u(x)$ and we assume that $u(0) < \bar{u}$, consider the expenditure function,

$$e(p, t) = \inf\{p \cdot x \mid x \in V(t)\},$$

and the Hicksian demand correspondence,

$$h(p, t) = \{x \in \mathbb{R}^N_+ \mid x \in V(t),\ p \cdot x = e(p, t)\},$$

where $V(t) = \{x \in \mathbb{R}^N_+ \mid u(x) \geq t\}$.

- If u is upper semi-continuous, then $h(p, t)$ is nonempty- and compact-valued and upper semi-continuous in p.
- If in addition, u is locally insatiable, then $h(p, t)$ is upper semi-continuous in (p, t) and $e(p, t)$ is continuous in (p, t).

  (Proposition 3.17)

Proposition 9.9

Assume that

1. u *is upper semi-continuous, and*
2. $h(\bar{p}, \bar{t}) = \{\bar{x}\}$.

Then e *is differentiable in* p *at* $(\bar{p}, \bar{t})$ *with*

$$\nabla_p e(\bar{p}, \bar{t}) = \bar{x}.$$

Proof

- By Proposition 3.17, the upper semi-continuity of u implies that $h(p, \bar{t})$ is nonempty-valued and upper semi-continuous in p.
- The function $f(x, p) = p \cdot x$ is differentiable in p, and $\nabla_p f(x, p) = x$ is continuous in (x, p).
- With $h(\bar{p}) = \{\bar{x}\}$, the conclusion follows from Corollary 9.5.

Proposition 9.10

Assume that

1. u *is locally insatiable and continuous,*

2. $h(\bar{p}, \bar{t}) = \{\bar{x}\}$, *where* $\bar{t} > u(0)$,

3. *for some* j *with* $\bar{x}_j > 0$ *and for some neighborhoods* X_j^0 *and* X_{-j}^0 *of* $\bar{x}_j$ *and* $\bar{x}_{-j}$ *in* $\mathbb{R}_+$ *and* $\mathbb{R}_+^{N-1}$, *respectively,* $\frac{\partial u}{\partial x_j}$ *exists on* $X_j^0 \times X_{-j}^0$ *and is continuous in* x *at* $\bar{x}$, *and*

4. $\frac{\partial u}{\partial x_j}(\bar{x}) \neq 0$ *for some* j *satisfying the condition in 3.*

Then e *is differentiable at* $(\bar{p}, \bar{t})$ *with*

$$\frac{\partial e}{\partial p_i}(\bar{p}, \bar{t}) = \bar{x}_i, \quad \frac{\partial e}{\partial t}(\bar{p}, \bar{t}) = \frac{\bar{p}_j}{\frac{\partial u}{\partial x_j}(\bar{x})},$$

for any j *satisfying the condition in 3.*

Proof

- By the upper semi-continuity and local insatiability of u, e is continuous in (p, t).
- By the continuity of u, $e(p, t)$ is a solution to the equation $v(p, w) - t = 0$ in w (which is unique by local insatiability), and $x(\bar{p}, \bar{w}) = h(\bar{p}, \bar{t}) = \{\bar{x}\}$, where $\bar{w} = e(\bar{p}, \bar{t})$. (See, e.g., Proposition 3.E.1 in MWG.)
- Therefore, combined with Assumptions 1 and 3, it follows from Proposition 9.8 that v is differentiable at $(\bar{p}, \bar{w})$.

Proof

- Combined with Assumption 4, it follows from a version of the Implicit Function Theorem that the solution function $e(p,t)$ to the equation $v(p,w) - t = 0$ in w is differentiable at $(\bar{p}, \bar{t})$ with

$$
\begin{aligned}
\frac{\partial e}{\partial p_i}(\bar{p}, \bar{t}) &= -\frac{\frac{\partial v}{\partial p_i}(\bar{p}, \bar{t})}{\frac{\partial v}{\partial w}(\bar{p}, \bar{t})} = \bar{x}_i, \\
\frac{\partial e}{\partial t}(\bar{p}, \bar{t}) &= -\frac{-1}{\frac{\partial v}{\partial w}(\bar{p}, \bar{t})} = \frac{\bar{p}_j}{u_{x_j}(\bar{x})},
\end{aligned}
$$

as claimed.

Remark

- The continuity of $\frac{\partial u}{\partial x_j}$ in x in Assumption 3 in Propositions 9.8 and 9.10 cannot be dropped.

  See Oyama and Takenawa, Example 5.1.

Concave Value Function

Let A be convex.

Proposition 9.11

Assume that

1. $X^*(q) \neq \emptyset$ *for all* $q \in A$,
2. *for all* $x \in X$, $f(x, \cdot)$ *is differentiable, and*
3. v *is concave.*

Then v *is differentiable at* $\bar{q}$ *with*

$$\frac{\partial v}{\partial q_s}(\bar{q}) = \frac{\partial f}{\partial q_s}(\bar{x}, \bar{q}), \qquad s = 1, \ldots, S$$

for any $\bar{x} \in X^*(\bar{q})$.

Remark

- If X is convex and f is concave in (x, q), then v is concave.