8. Optimization

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Unconstrained Maximization Problem

Let $X \subset \mathbb{R}^N$ be a nonempty set.

Definition 8.1

For a function $f\colon X \to \mathbb{R}$,

- $\bar{x} \in X$ is a *(strict) local maximizer* of f if
  there exists an open neighborhood $A \subset X$ of $\bar{x}$ relative to X
  such that $f(\bar{x}) \geq f(x)$ for all $x \in A$
  ($f(\bar{x}) > f(x)$ for all $x \in A$ with $x \neq \bar{x}$);
- $\bar{x} \in X$ is a *maximizer* (or *global maximizer*) of f if
  $f(\bar{x}) \geq f(x)$ for all $x \in X$.

(Local and global minimizers are defined analogously.)

First-Order Condition for Optimality

Let $X \subset \mathbb{R}^N$ be a nonempty set.

Proposition 8.1

For $f\colon X \to \mathbb{R}$, if

- $\bar{x} \in X$ *is a local maximizer or local minimizer of f,*
- $\bar{x} \in \operatorname{Int} X$, *and*
- f *is differentiable at $\bar{x}$,*

then $\nabla f(\bar{x}) = 0$.

Proof

Apply the FOC for the one variable case to $f(x_i, \bar{x}_{-i})$ for each $i = 1, \ldots, N$.

Second-Order Condition for Optimality

Let $X \subset \mathbb{R}^N$ be a nonempty set.

Proposition 8.2

For $f\colon X \to \mathbb{R}$, suppose that $\bar{x} \in \operatorname{Int} X$ and that f is differentiable on $\operatorname{Int} X$ and ∇f is differentiable at $\bar{x}$.

1. *If $\bar{x}$ is a local maximizer of f, then $D^2 f(\bar{x})$ is negative semi-definite.*

2. *If $\nabla f(\bar{x}) = 0$ and $D^2 f(\bar{x})$ is negative definite, then $\bar{x}$ is a strict local maximizer of f.*

Proof

1.

- Fix any $z \in \mathbb{R}^N$, $z \neq 0$.

  Let $h(\alpha) = f(\bar{x} + \alpha z) - f(\bar{x})$
  (where $\alpha \in \mathbb{R}$ is sufficiently close to 0).

  Note that h is differentiable and h' is differentiable at $\alpha = 0$.

- Recall that $h''(\alpha) = z \cdot D^2 f(\bar{x} + \alpha z) z$.

- If $\bar{x}$ is a local maximizer of f,
  then $\alpha = 0$ is a local maximizer of h.

- If $h''(0) > 0$, then $\alpha = 0$ would be a strict local minimizer.

- Thus, $h''(0) \leq 0$, or $z \cdot D^2 f(\bar{x}) z \leq 0$.

2.

- Suppose that $\nabla f(\bar{x}) = 0$ and $D^2 f(\bar{x})$ is negative definite.
- Since $u \cdot D^2 f(\bar{x}) u$ is continuos in u and since $\{u \in \mathbb{R}^N \mid \|u\| = 1\}$ is compact, it follows from the assumption of negative definiteness and the Extreme Value Theorem that there is some $\varepsilon > 0$ such that

$$\frac{1}{2} u \cdot D^2 f(\bar{x}) u + \varepsilon < 0 \text{ for all } u \in \mathbb{R}^N \text{ such that } \|u\| = 1.$$

- Since $\nabla f(\bar{x}) = 0$, by Taylor's Theorem we can take a sufficiently small $\delta > 0$ such that

$$0 < \|z\| < \delta \Rightarrow \frac{f(\bar{x} + z) - f(\bar{x})}{\|z\|^2} - \frac{\frac{1}{2} z \cdot D^2 f(\bar{x}) z}{\|z\|^2} \leq \varepsilon.$$

- Now take any $x \in B_\delta(\bar{x})$, $x \neq \bar{x}$.

  Then,

$$\frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|^2} \leq \frac{1}{2} \frac{x - \bar{x}}{\|x - \bar{x}\|} \cdot D^2 f(\bar{x}) \frac{x - \bar{x}}{\|x - \bar{x}\|} + \varepsilon < 0,$$

  where the last inequality follows from the choice of ε.

  Thus, $f(x) < f(\bar{x})$.

Concave Functions

Let $X \subset \mathbb{R}^N$ be a nonempty convex set.

Proposition 8.3

For $f\colon X \to \mathbb{R}$, suppose that $\bar{x} \in \operatorname{Int} X$ and f is differentiable at $\bar{x}$.

- *Suppose that f is concave.*

  If $\nabla f(\bar{x}) = 0$, then $\bar{x}$ is a global maximizer of f.

- *Suppose that f is strictly concave.*

  If $\nabla f(\bar{x}) = 0$, then $\bar{x}$ is a unique global maximizer of f.

Proof

- Take any $x \in X$, $x \neq \bar{x}$.
- If f is concave, then we have

  $$f(x) \leq f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x}),$$

  with a strict inequality if f is strictly concave.
- Thus, if $\nabla f(\bar{x}) = 0$, we have $f(x) \leq f(\bar{x})$ if f is concave, and $f(x) < f(\bar{x})$ if f is strictly concave.

Equality Constrained Maximization Problem

Let $X \subset \mathbb{R}^N$ be a nonempty open set, and $f, g_1, \dots, g_M \colon X \to \mathbb{R}$, where $M < N$.

Consider the maximization problem:

$$\begin{aligned} \max_x \quad & f(x) \\ \text{s.t.} \quad & g_1(x) = 0 \\ & \vdots \\ & g_M(x) = 0. \end{aligned} \tag{P}$$

- Write $g \colon X \to \mathbb{R}^M$, $x \mapsto (g_1(x), \dots, g_M(x))$, and $C = \{x \in X \mid g(x) = 0\}$.
- $\bar{x} \in C$ is a local (global, resp.) constrained maximizer of (P) if it is a local (global, resp.) maximizer of $f|_C$.

First-Order Condition for Optimality

Proposition 8.4

Suppose that

- $\bar{x} \in C$ *is a local constrained maximizer of* (P)*;*
- f *is differentiable at* $\bar{x}$*;*
- $g_1, \ldots, g_M$ *are* C^1 *on a neighborhood of* $\bar{x}$*; and*
- $\operatorname{rank} Dg(\bar{x}) = M$ *("constraint qualification").*

Then there exist unique $(\bar{\lambda}_1, \ldots, \bar{\lambda}_M) \in \mathbb{R}^M$ *(Lagrange multipliers) such that*

$$\nabla f(\bar{x}) = \sum_{m=1}^{M} \bar{\lambda}_m \nabla g_m(\bar{x}).$$

Expression with Lagrangian

- Let $L\colon X \times \mathbb{R}^M \to \mathbb{R}$ be defined by

$$L(x, \lambda) = f(x) - \sum_{m=1}^{M} \lambda_m g_m(x).$$

- Then the FOC is:

  there exists $\bar{\lambda} \in \mathbb{R}^M$ such that

$$\begin{aligned} \frac{\partial L}{\partial x_n}(\bar{x}, \bar{\lambda}) = 0, &\quad n = 1, \dots, N, \\ \frac{\partial L}{\partial \lambda_m}(\bar{x}, \bar{\lambda}) = 0, &\quad m = 1, \dots, M, \end{aligned}$$

  or

$$\nabla L(\bar{x}, \bar{\lambda}) = 0.$$

Proof

- Let $\bar{x} \in C$ be a local constrained maximizer.

  By assumption $Dg(\bar{x}) \in \mathbb{R}^{M \times N}$ has rank M.
- Without loss of generality, assume that the first M columns of $Dg(\bar{x})$ are linearly independent.

  Write $x = (p, q)$, where $p \in \mathbb{R}^M$ and $q \in \mathbb{R}^{N-M}$.
- By the Implicit Function Theorem, the equation $g(p, q) = 0$ is locally solved as $p = \eta(q)$, where

$$D\eta(\bar{q}) = -[D_p g(\bar{p}, \bar{q})]^{-1} D_q g(\bar{p}, \bar{q}).$$

- Consider the unconstrained maximization problem $F(q) = f(\eta(q), q)$, where $\bar{q}$ is a local maximizer.

- By the FOC $DF(\bar{q}) = 0$, we have

$$\begin{aligned} 0 &= D_q f(\eta(q), q)|_{q=\bar{q}} \\ &= D_p f(\bar{x}) D\eta(\bar{q}) + D_q f(\bar{x}) \\ &= -D_p f(\bar{x})[D_p g(\bar{x})]^{-1} D_q g(\bar{x}) + D_q f(\bar{x}). \end{aligned}$$

- Let $\bar{\lambda}^{\mathrm{T}} = D_p f(\bar{x})[D_p g(\bar{x})]^{-1}$, where $\bar{\lambda} \in \mathbb{R}^M$.
- Then we have

$$D_p f(\bar{x}) = \bar{\lambda}^{\mathrm{T}} D_p g(\bar{x}), \quad D_q f(\bar{x}) = \bar{\lambda}^{\mathrm{T}} D_q g(\bar{x}),$$

or

$$\nabla f(\bar{x}) = Dg(\bar{x})^{\mathrm{T}} \bar{\lambda} = \sum_{m=1}^{M} \bar{\lambda}_m \nabla g_m(\bar{x}).$$

Second-Order Condition for Optimality

Proposition 8.5

Suppose that $f, g_1, \dots, g_M$ *are of* C^2 *class,* $\bar{x} \in C$*, and* $\operatorname{rank} Dg(\bar{x}) = M$*.*
Denote $W = \{z \in \mathbb{R}^N \mid Dg(\bar{x})z = 0\}$*.*

1. *If* $\bar{x}$ *is a local constrained maximizer of* (P)*, then* $D_x^2 L(\bar{x}, \bar{\lambda})$ *is negative semi-definite on* W*, where* $\bar{\lambda} \in \mathbb{R}^M$ *is such that* $\nabla L(\bar{x}, \bar{\lambda}) = 0$*.*

2. *If there exists* $\bar{\lambda} \in \mathbb{R}^M$ *such that* $\nabla L(\bar{x}, \bar{\lambda}) = 0$ *and* $D_x^2 L(\bar{x}, \bar{\lambda})$ *is negative definite on* W*, then* $\bar{x}$ *is a strict local constrained maximizer of* (P)*.*

Inequality Constrained Maximization Problem

Let $X \subset \mathbb{R}^N$ be a nonempty open set, and $f, g_1, \dots, g_M, h_1, \dots, h_K \colon X \to \mathbb{R}$, where $M < N$.

Consider the maximization problem:

$$
\begin{aligned}
\max_x \quad & f(x) && \text{(P)} \\
\text{s. t.} \quad & g_1(x) = 0 \\
& \quad \vdots \\
& g_M(x) = 0 \\
& h_1(x) \leq 0 \\
& \quad \vdots \\
& h_K(x) \leq 0.
\end{aligned}
$$

- Write $C = \{x \in X \mid g(x) = 0,\ h(x) \leq 0\}$.
- $\bar{x} \in C$ is a local (global, resp.) constrained maximizer of (P) if it is a local (global, resp.) maximizer of $f|_C$.

First-Order Condition for Optimality (KKT Conditions)

For $x \in C$, write $\mathcal{I}(x) = \{k \mid h_k(x) = 0\}$.

Proposition 8.6

Suppose that

- $\bar{x} \in C$ *is a local constrained maximizer of* (P)*;*
- $f, h_1, \ldots, h_K$ *are differentiable at* $\bar{x}$*;*
- $g_1, \ldots, g_M$ *are* C^1 *on a neighborhood of* $\bar{x}$*; and*
- $\nabla g_1(\bar{x}), \ldots, \nabla g_M(\bar{x})$ *and* $\nabla h_k(\bar{x})$, $k \in \mathcal{I}(\bar{x})$, *are linearly independent ("constraint qualification").*

Then there exist $\bar{\mu}_1, \ldots, \bar{\mu}_M \in \mathbb{R}$ *and* $\bar{\lambda}_1, \ldots, \bar{\lambda}_K \in \mathbb{R}$ *such that*

(i) $\nabla f(\bar{x}) = \sum_{m=1}^{M} \bar{\mu}_m \nabla g_m(\bar{x}) + \sum_{k=1}^{K} \bar{\lambda}_k \nabla h_k(\bar{x})$, *and*

(ii) $\bar{\lambda}_k \geq 0$ *and* $\bar{\lambda}_k h_k(\bar{x}) = 0$ *for each* $k = 1, \ldots, K$*.*

- "$\bar{\lambda}_k h_k(\bar{x}) = 0$" is called the *complementarity condition*.
- It says: $\bar{\lambda}_k = 0$ for all $k \notin \mathcal{I}(\bar{x})$,

  where $\mathcal{I}(x) = \{k \mid h_k(x) = 0\}$.

Example 1

Let $X = \mathbb{R}$.

Consider

$$\max_{x \in [0,1]} f(x),$$

or

$$\begin{aligned} \max_{x} \quad & f(x) \\ \text{s. t.} \quad & h_1(x) = -x \leq 0 \\ & h_2(x) = x - 1 \leq 0. \end{aligned}$$

- If $\bar{x} \in [0, 1]$ is a local constrained maximizer, then clearly we have:
  1. if $\bar{x} \in (0, 1)$, then $f'(\bar{x}) = 0$,
  2. if $\bar{x} = 0$, then $f'(\bar{x}) \leq 0$,
  3. if $\bar{x} = 1$, then $f'(\bar{x}) \geq 0$.

Example 1

- Let

$$L(x,\lambda) = f(x) - \lambda_1(-x) - \lambda_2(x-1).$$

- The KKT conditions are:

$$L_x(x,\lambda) = f'(x) + \lambda_1 - \lambda_2 = 0 \iff f'(x) = -\lambda_1 + \lambda_2,$$
$$\lambda_1 \geq 0,\ \lambda_1(-x) = 0,$$
$$\lambda_2 \geq 0,\ \lambda_2(x-1) = 0.$$

- By these,

1. if $-\bar{x} < 0$ and $\bar{x} - 1 < 0$, then $\lambda_1 = \lambda_2 = 0$, so $f'(\bar{x}) = 0$,
2. if $-\bar{x} = 0$ and $\bar{x} - 1 < 0$, then $\lambda_2 = 0$, so $f'(\bar{x}) = -\lambda_1 \leq 0$,
3. if $-\bar{x} < 0$ and $\bar{x} - 1 = 0$, then $\lambda_1 = 0$, so $f'(\bar{x}) = \lambda_2 \geq 0$.

Example 1

- To see why we have $\lambda_k \geq 0$, suppose that $\bar{x}$ satisfies the constraint $h_k(x) \leq 0$ with "$=$" (i.e., $h_k(\bar{x}) = 0$).
- For $z \approx 0$, $f(\bar{x} + z) \approx f(\bar{x}) + f'(\bar{x})z$ and $h_k(\bar{x} + z) \approx h_k'(\bar{x})z$.
- If $f'(\bar{x}) > 0$,
  then for small $\varepsilon > 0$, $\bar{x} + \varepsilon$ has to violate the constraint,
  for which we have to have $h_k'(\bar{x}) \geq 0$.

  (Constraint qualification implies that $h_k'(\bar{x}) \neq 0$.)
- If $f'(\bar{x}) < 0$,
  then for small $\varepsilon > 0$, $\bar{x} - \varepsilon$ has to violate the constraint,
  for which we have to have $h_k'(\bar{x}) \leq 0$.
- In these cases, we have $f'(\bar{x}) = \lambda_k h_k'(\bar{x})$ with $\lambda_k > 0$.
- It is possible that $f'(\bar{x}) = 0$, so it may be the case that $\lambda_k = 0$.

Example 2

For $p \gg 0$ and $w > 0$, consider

$$\begin{aligned} \max_{x} \quad & u(x) \\ \text{s. t.} \quad & p \cdot x - w \leq 0 \\ & -x_1 \leq 0, \dots, -x_N \leq 0. \end{aligned}$$

- The KKT conditions: $\bar{x} \neq 0$,

$$\begin{aligned} & \nabla u(\bar{x}) = \mu p - \sum_{n=1}^{N} \lambda_n e_n, \\ & \mu \geq 0, \ \mu(p \cdot \bar{x} - w) = 0, \\ & \lambda_n \geq 0, \ \lambda_n(-\bar{x}_n) = 0 \quad (n = 1, \dots, N). \end{aligned}$$

- These can be written as

$$\begin{aligned} & \frac{\partial u}{\partial x_n}(\bar{x}) \leq \mu p_n, \quad \text{with equality if } \bar{x}_n > 0 \quad (n = 1, \dots, N), \\ & \mu \geq 0, \ \mu(p \cdot \bar{x} - w) = 0. \end{aligned}$$

Example 2

Let $N = 2$.

- Suppose that $\bar{x} = (w/p_1, 0)$.
- First, we have to have $\dfrac{\partial u}{\partial x_1}(\bar{x}) \geq 0$.

  So we have $\dfrac{\partial u}{\partial x_1}(\bar{x}) = \lambda p_1$ for some $\lambda \geq 0$.
- Thus, we have to have $\dfrac{\partial u}{\partial x_2}(\bar{x}) \leq \lambda p_2$.

  (Draw a picture.)

Proof of Proposition 8.6

Case with no equality constraint.

- Note that for any $z \in \mathbb{R}^N$,

$$
\begin{aligned}
f(\bar{x} + tz) &= f(\bar{x}) + (\nabla f(\bar{x}) \cdot z)t + o(t), \\
h_k(\bar{x} + tz) &= (\nabla h_k(\bar{x}) \cdot z)t + o(t) \quad \text{for all } k \in \mathcal{I}(\bar{x}).
\end{aligned}
$$

- Since $\bar{x}$ is a local constrained maximizer, there is no $z \in \mathbb{R}^N$ such that $\nabla f(\bar{x}) \cdot z > 0$ and $\nabla h_k(\bar{x}) \cdot z < 0$ for all $k \in \mathcal{I}(\bar{x})$, or $\begin{pmatrix} Df(\bar{x}) \\ -Dh_{\mathcal{I}}(\bar{x}) \end{pmatrix} z \gg 0$.

- Thus, by Gordan's Theorem, there exist $\lambda_0, \lambda_k \geq 0$, $k \in \mathcal{I}(\bar{x})$, such that

$$
\lambda_0 \nabla f(\bar{x}) - \sum_{k \in \mathcal{I}(\bar{x})} \lambda_k \nabla h_k(\bar{x}) = 0, \quad (\lambda_0, \lambda_k)_{k \in \mathcal{I}(\bar{x})} \neq 0.
$$

- By the constraint qualification, $\lambda_0 \neq 0$, so normalize $\lambda_0 \equiv 1$.

Proof of Proposition 8.6

Case with inequality and equality constraints.

- We show that there is no $z \in \mathbb{R}^N$ such that $Df(\bar{x})z > 0$, $-Dh_{\mathcal{I}}(\bar{x})z \gg 0$, and $Dg(\bar{x})z = 0$.
- Write $x = (p, q)$, where $p \in \mathbb{R}^M$ and $q \in \mathbb{R}^{N-M}$.

  $g(p, q) = 0$ is solved locally around $\bar{x} = (\bar{p}, \bar{q})$ as $p = \eta(q)$, where $D\eta(\bar{q}) = -[D_p g(\bar{x})]^{-1} D_q g(\bar{x})$.
- Suppose that $Dg(\bar{x})z = 0$, or $D_p g(\bar{x})u + D_q g(\bar{x})v = 0$ so that $u = -[D_p g(\bar{x})]^{-1} D_q g(\bar{x})v = D\eta(\bar{q})v$, where $z = (u, v)$.
- Let $x(t) = (\eta(\bar{q} + tv), \bar{q} + tv)$.

  Then
  $$Dx(0) = \begin{pmatrix} D\eta(\bar{q})v \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} = z.$$

- Now we have

$$\begin{aligned} f(x(t)) &= f(\bar{x}) + (\nabla f(\bar{x}) \cdot Dx(0))t + o(t) \\ &= f(\bar{x}) + (\nabla f(\bar{x}) \cdot z)t + o(t), \\ h_k(\bar{x} + tz) &= (\nabla h_k(\bar{x}) \cdot Dx(0))t + o(t) \\ &= (\nabla h_k(\bar{x}) \cdot z)t + o(t) \quad \text{for all } k \in \mathcal{I}(\bar{x}). \end{aligned}$$

- Since $\bar{x}$ is a local constrained maximizer, we cannot have $\nabla f(\bar{x}) \cdot z > 0$ and $h_k(\bar{x}) \cdot z < 0$ for all $k \in \mathcal{I}(\bar{x})$.
- I.e., $\nexists\, z \in \mathbb{R}^N$ such that $\begin{pmatrix} Df(\bar{x}) \\ -Dh_{\mathcal{I}}(\bar{x}) \end{pmatrix} z \gg 0$ and $Dg(\bar{x})z = 0$.
- Thus, by Motzkin's Theorem, there exist $(\lambda_0, \lambda_{\mathcal{I}}) \gneq 0$ and μ such that

$$\begin{pmatrix} \lambda_0 & \lambda_{\mathcal{I}}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} Df(\bar{x}) \\ -Dh_{\mathcal{I}}(\bar{x}) \end{pmatrix} + \mu^{\mathrm{T}} Dg(\bar{x}) = 0.$$

- By the constraint qualification, $\lambda_0 \neq 0$; so normalize $\lambda_0 \equiv 1$.

Second-Order Condition for Optimality

Proposition 8.7

Suppose that $f, g_1, \dots, g_M, h_1, \dots, h_K$ *are of* C^2 *class,* $\bar{x} \in C$, *and* $\nabla g_1(\bar{x}), \dots, \nabla g_M(\bar{x})$ *and* $\nabla h_k(\bar{x})$, $k \in \mathcal{I}$, *are linearly independent. If*

- *there exist* $\bar{\mu}_1, \dots, \bar{\mu}_M \in \mathbb{R}$ *and* $\bar{\lambda}_1, \dots, \bar{\lambda}_K \in \mathbb{R}$ *such that the KKT conditions hold, and*
- $D_x^2 L(\bar{x}, \bar{\lambda})$ *is negative definite on* W, *where*

$$\begin{aligned} W = \{z \in \mathbb{R}^N \mid \nabla g_m(\bar{x}) \cdot z = 0 \text{ for all } m = 1, \dots, M, \\ \nabla h_k(\bar{x}) \cdot z = 0 \text{ for all } k \in \tilde{\mathcal{I}}\}, \end{aligned}$$

  and $\tilde{\mathcal{I}} = \{k \mid \bar{\lambda}_k > 0\}$,

then $\bar{x}$ *is a strict local constrained maximizer of* (P).

Quasi-Concavity/Convexity

Proposition 8.8

Suppose that $f, h_1, \ldots, h_K$ *are of* C^1 *class and* $g_1, \ldots, g_M$ *are affine (i.e.,* $g_m(x) = a^m \cdot x + b^m$*), and* $\bar{x} \in C$*.*
Suppose that

1. $f(x') > f(x) \Longrightarrow \nabla f(x) \cdot (x' - x) > 0$*, and*
2. *for all* $k = 1, \ldots, K$*,*
   $h_k(x') \leq h_k(x) \Longrightarrow \nabla h_k(x) \cdot (x' - x) \leq 0$*;*

Then if $\bar{x}$ *satisfies the KKT conditions for some* $\mu_1, \ldots, \mu_M, \lambda_1, \ldots, \lambda_K \in \mathbb{R}$*, then* $\bar{x}$ *is a global constrained maximizer of* (P)*.*

Proof

- Let $\bar{x} \in C$ satisfy the KKT conditions, and take any $x' \in C$ with $x' \neq \bar{x}$.
- If $\lambda_k > 0$, then $h_k(\bar{x}) = 0$. With $h_k(x') \leq 0$, we have $h_k(x') \leq h_k(\bar{x})$.
- Therefore, by Condition 2, we have $\nabla h_k(\bar{x}) \cdot (x' - \bar{x}) \leq 0$ whenever $\lambda_k > 0$.
- It follows from the KKT conditions that

$$\begin{aligned}\nabla f(\bar{x}) \cdot (x' - \bar{x}) &= \sum \mu_m a^m \cdot (x' - \bar{x}) + \sum \lambda_k \nabla h_k(\bar{x}) \cdot (x' - \bar{x}) \\ &\leq 0.\end{aligned}$$

- Hence, by Condition 1, we have $f(x') \leq f(\bar{x})$.

Remarks

- Condition 2 $\iff h_k$ is quasi-convex.
- When Condition 1 holds, f is called *pseudo-concave*.
- f: strictly quasi-concave and $\nabla f(x) \neq 0$ for all x
  $\Rightarrow f$: pseudo-concave
  $\Rightarrow f$: quasi-concave

Quasi-Concavity

Proposition 8.9

Let $C \subset \mathbb{R}^N$ be a nonempty convex set.
Suppose that $f : C \to \mathbb{R}$ is strictly quasi-concave, and consider the maximization problem

$$\max_{x \in C} f(x).$$

If $\bar{x} \in C$ is a local maximizer, then it is a unique global maximizer.