1. Real Numbers

Daisuke Oyama

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Notations

- $\mathbb{N} = \{1, 2, 3, \ldots\}$: the set of natural numbers (often 0 included)
- $\mathbb{Z}$: the set of integers
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{R}$: the set of real numbers

Properties of $\mathbb{R}$

(We do not discuss how to construct real numbers.)

1. Binary operations, addition $+$ and multiplication $\cdot$, are defined (commutative, associative, distributive).
2. Complete ordering $\leq$ is defined (complete, transitive, antisymmetric).
3. Every nonempty subset of $\mathbb{R}$ that has an upper bound (or is bounded above) has a least upper bound (or supremum).

   $\cdots$ "Axiom of Real Numbers"

- Property 3 is the property that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

  $\mathbb{Q}$ does not satisfy property 3.

  Example:
  $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$ has no least upper bound in $\mathbb{Q}$.

Maximum/Minimum

Let A be a subset of $\mathbb{R}$.

- $x \in \mathbb{R}$ is the *greatest element* of A or the *maximum* of A, denoted $\max A$, if
  - $x \in A$, and
  - $a \leq x$ for all $a \in A$.
- $x \in \mathbb{R}$ is the *least element* of A or the *minimum* of A, denoted $\min A$, if
  - $x \in A$, and
  - $x \leq a$ for all $a \in A$.

Upper/Lower Bounds, Supremum/Infimum

Let A be a nonempty subset of $\mathbb{R}$.

- $x \in \mathbb{R}$ is an upper bound of A if $a \leq x$ for all $a \in A$.
- $x \in \mathbb{R}$ is the supremum of A, denoted $\sup A$, if it is the least upper bound of A, i.e.,
  - x is an upper bound of A, and
  - if y is an upper bound of A, then $x \leq y$.

Likewise,

- $x \in \mathbb{R}$ is a lower bound of A if $x \leq a$ for all $a \in A$.
- $x \in \mathbb{R}$ is the infimum of A, denoted $\inf A$, if it is the greatest lower bound of A.

Recap

Let A be a nonempty subset of $\mathbb{R}$.

- $\sup A$ = least upper bound of A
- $\inf A$ = greatest lower bound of A

- Property 3 says that $\sup A$ exists if A is bounded above.
- Property 3 implies that $\inf A$ exists if A is bounded below.

Example

Let $A = (0, 1] = \{a \in \mathbb{R} \mid 0 < a \leq 1\}$.

- $\max A = \sup A = 1$.
- $\min A$ does not exist, while $\inf A = 0$.

Characterization of $\sup$ and $\inf$

Proposition 1.1

1. $x = \sup A$ *if and only if*
   (i) $a \leq x$ *for all* $a \in A$*, and*
   (ii) *for all* $\varepsilon > 0$*, there exists* $a \in A$ *such that* $x - \varepsilon < a$*.*
2. $x = \inf A$ *if and only if*
   (i) $x \leq a$ *for all* $a \in A$*, and*
   (ii) *for all* $\varepsilon > 0$*, there exists* $a \in A$ *such that* $a < x + \varepsilon$*.*

Derived Properties of $\mathbb{R}$

Proposition 1.2 (Archimedean Property)

$\mathbb{N}$ *is unbounded above (i.e.,* $\mathbb{N}$ *has no upper bound) in* $\mathbb{R}$*.*

Proof

- Assume that $\mathbb{N}$ is bounded above.
- Then $\alpha = \sup \mathbb{N}$ exists by the Axiom of Real Numbers.
- By the definition of $\sup$, there is some $n \in \mathbb{N}$ such that $\alpha - 1 < n$.
- Then $\alpha < n + 1$, where $n + 1 \in \mathbb{N}$.
- This contradicts the assumption that $\alpha = \sup \mathbb{N}$.

Derived Properties of $\mathbb{R}$

Proposition 1.3 (Denseness of $\mathbb{Q}$ in $\mathbb{R}$)

For any $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof

- We only consider the case where $0 \leq a < b$.
- By the Archimedean Property, there is some $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$.
- Let $m \in \mathbb{N}$ be such that $m - 1 \leq na < m$.
  $\Rightarrow a < \frac{m}{n}$.
- Then

$$nb = na + n(b - a) > (m - 1) + 1 = m.$$

  $\Rightarrow \frac{m}{n} < b$.
- So let $r = \frac{m}{n}$.

Convergence in $\mathbb{R}$

- A *sequence* in $\mathbb{R}$ is a function from $\mathbb{N}$ to $\mathbb{R}$.

  A sequence is denoted by $\{x^m\}_{m=1}^{\infty}$, or simply $\{x^m\}$, or x^m.

- A sequence $\{x^m\}_{m=1}^{\infty}$ *converges* to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists a natural number M such that

  $$|x^m - \alpha| < \varepsilon \text{ for all } m \geq M.$$

  In this case, we write

  $$\lim_{m\to\infty} x^m = \alpha \qquad \text{or} \qquad x^m \to \alpha \text{ (as } m \to \infty).$$

- α is called the *limit* of $\{x^m\}_{m=1}^{\infty}$.

  (If $x^m \to \alpha$ and $x^m \to \beta$, then $\alpha = \beta$.)

- A sequence that converges to some $\alpha \in \mathbb{R}$ is said to be *convergent*.

Example

Let $x^m = \dfrac{1}{m}$.

Then $\lim_{m\to\infty} x^m = 0$.

- Take any $\varepsilon > 0$.
- By the Archimedean Property, we can take a natural number $M > \frac{1}{\varepsilon}$.
- Then for all $m \geq M$, we have

$$|x^m - 0| = \frac{1}{m} \leq \frac{1}{M} < \varepsilon.$$

Derived Properties of $\mathbb{R}$

Proposition 1.4 (Convergence of Monotone Sequences)

Every monotone increasing (decreasing, resp.) sequence $\{x^m\}$ in $\mathbb{R}$ that is bounded above (below, resp.) is convergent, where the limit equals $\sup\{x^m\}$ ($\inf\{x^m\}$, resp.).

Proof

- For a monotone increasing and bounded sequence $\{x^m\}$, let $A = \{x^1, x^2, x^3, \dots\}$.
- Since A is bounded above,
  $\alpha = \sup A$ exists by the Axiom of Real Numbers.
  $\Rightarrow x^m \le \alpha$ for all $m \in \mathbb{N}$.
- Fix any $\varepsilon > 0$.
- By the definition of $\sup A$, there exists $M \in \mathbb{N}$ such that $\alpha - \varepsilon < x^M$.
- Since x^m is increasing, $\alpha - \varepsilon < x^m$ for all $m \ge M$.
- Therefore, we have $|x^m - \alpha| < \varepsilon$ for all $m \ge M$.

Derived Properties of $\mathbb{R}$

Write $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ (called a *closed interval*).

Proposition 1.5 (Nested Intervals Theorem)

Suppose that closed intervals $I^m = [a^m, b^m]$*, where* $a^m \leq b^m$*, satisfy* $I^m \supset I^{m+1}$*,* $m = 1, 2, \ldots$*. Then,* $\bigcap_{m=1}^{\infty} I^m \neq \emptyset$*.*
If $b^m - a^m \to 0$ *as* $m \to \infty$*, then for some* $\alpha \in \mathbb{R}$*,*
$\lim_{m\to\infty} a^m = \lim_{m\to\infty} b^m = \alpha$ *and* $\bigcap_{m=1}^{\infty} I^m = \{\alpha\}$*.*

Proof

By Convergence of Bounded Monotone Sequences.

Derived Properties of $\mathbb{R}$

Proposition 1.6 (Bolzano-Weierstrass Theorem)

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

- For a sequence $\{x^m\}_{m=1}^{\infty}$ and a strictly increasing function $m(k)$ from $\mathbb{N}$ to $\mathbb{N}$, the sequence $\{x^{m(1)}, x^{m(2)}, \ldots\}$ (denoted $\{x^{m(k)}\}_{k=1}^{\infty}$) is called a *subsequence* of $\{x^m\}_{m=1}^{\infty}$.

Proof (1/2)

- Let $\{x^m\}$ be a bounded sequence, and let $I^1 = [a^1, b^1]$ be such that $x^m \in I^1$ for all $m \in \mathbb{N}$.
- Either $\{m \in \mathbb{N} \mid x^m \in [a^1, (a^1 + b^1)/2]\}$ or $\{m \in \mathbb{N} \mid x^m \in [(a^1 + b^1)/2, b^1]\}$ (or both) contains infinitely many elements of $\{x^m\}$.

  Let $I^2 = [a^2, b^2]$ be such an interval
  (let $I^2 = [a^1, (a^1 + b^1)/2]$ if both contain infinitely many elements).
- Repeat this procedure, and we have a sequence of closed intervals $I^1 \supset I^2 \supset I^3 \supset \cdots$, which satisfies
  $b^m - a^m = 2^{-(m-1)}(b^1 - a^1) \to 0$ as $m \to \infty$
  by the Archimedean Property.
- By the Nested Intervals Theorem,
  $\lim_{m\to\infty} a^m = \lim_{m\to\infty} b^m = \alpha$ for some $\alpha \in \mathbb{R}$.

Proof (2/2)

- Define a subsequence $\{x^{m(k)}\}$ as follows:
  - Let $m(1) = 1$.
  - Pick any x^m from I^2 with $m > m(1)$, and let $m(2) = m$.
  - $\dots$
  - Pick any x^m from I^k with $m > m(k-1)$, and let $m(k) = m$.
  - $\dots$

  Then, since $a^k \leq x^{m(k)} \leq b^k$ for all k and $\lim_{k\to\infty} a^k = \lim_{k\to\infty} b^k = \alpha$, we have $x^{m(k)} \to \alpha$ as $k \to \infty$.

Derived Properties of $\mathbb{R}$

- A sequence $\{x^m\}_{m=1}^{\infty}$ is a *Cauchy sequence* if for any $\varepsilon > 0$, there exists a natural number M such that

$$|x^m - x^n| < \varepsilon \text{ for all } m, n \geq M.$$

- A Cauchy sequence is bounded.
- A convergent sequence is a Cauchy sequence.

Proposition 1.7 (Completeness of $\mathbb{R}$)
Every Cauchy sequence in $\mathbb{R}$ is convergent.

Proof
By the Bolzano-Weierstrass Theorem.

Derived Properties of $\mathbb{R}$

Proposition 1.8 (Decimal Representation of Real Numbers)

Fix any $N \in \mathbb{N}$ with $N \geq 2$.
For any $x \in \mathbb{R}$, there exists a sequence $\{k_m\}$ with $k_m = 0, 1, \ldots, N-1$ such that the sequence

$$a_m = [x] + \frac{k_1}{N} + \frac{k_2}{N^2} + \cdots + \frac{k_m}{N^m} \qquad (*)$$

converges to x as $m \to \infty$.
Conversely, a sequence $\{a_m\}$ of the form $(*)$ converges to some real number.

Derived Properties of $\mathbb{R}$

- If $A \subset \mathbb{R}$ is a *closed set*, then it has the following property:

  for any convergent sequence $\{x^m\}$ in A,
  we have $\lim_{m\to\infty} x^m \in A$.

  (Closed sets will be formally defined next class.)

Proposition 1.9 (Connectedness of $\mathbb{R}$)

Let $A, B \subset \mathbb{R}$ be nonempty closed sets.
If $\mathbb{R} = A \cup B$, then $A \cap B \neq \emptyset$.

Proof (1/2)

- Pick any $a \in A$ and $b \in B$.

  Assume without loss of generality that $a < b$.

- Let $A^- = \{x \in A \mid x \leq b\}$.

- $A^- \neq \emptyset$ since $a \in A^-$, and A^- is bounded above by b.

  Therefore, $a^* = \sup A^-$ exists by the Axiom of Real Numbers, where $a^* \leq b$.

- By the definition of $\sup A^-$, for any $m \in \mathbb{N}$ there is some $a^m \in A^-$ ($\subset A$) such that $a^* - \frac{1}{m} < a^m \leq a^*$.

  By construction, a^m converges to a^* as $m \to \infty$.

- Therefore, $a^* \in A$ since A is closed.

- If $a^* = b$, then we have $a^* = b \in B$.

Proof (2/2)

- Suppose that $a^* < b$.
- For each $m \in \mathbb{N}$, let $b^m = a^* + \frac{b-a^*}{m}$, where $a^* < b^m \leq b$.
- By the definition of $\sup A^-$, $b^m \notin A$.

  Therefore, $b^m \in B$ since $\mathbb{R} = A \cup B$.
- By construction, b^m converges to a^* as $m \to \infty$.
- Therefore, $a^* \in B$ since B is closed.

Remark

Any nonempty interval I in $\mathbb{R}$ has the same property:

Let $A, B \subset I$ be nonempty closed sets (relative to I).
If $I = A \cup B$, then $A \cap B \neq \emptyset$.

Cardinality of $\mathbb{R}$

For a function (or mapping) $f\colon X \to Y$,

- f is *one-to-one* (or an *injection*) if
  $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
- f is *onto* (or a *surjection*) if
  for any $y \in Y$, there exists $x \in X$ such that $y = f(x)$.
- f is a *bijection* if it is one-to-one and onto.
- If $f\colon X \to Y$ and $g\colon Y \to Z$ are one-to-one (onto, resp.),
  then $g \circ f\colon X \to Z$ is one-to-one (onto, resp.).

Proposition 1.10

1. *There is an onto mapping from $\mathbb{N}$ to $\mathbb{Z}$.*
2. *There is an onto mapping from $\mathbb{Z}$ to $\mathbb{Q}$.*
3. *There is an onto mapping from $(0, 1)$ to $\mathbb{R}$.*
4. *There is no onto mapping from $\mathbb{N}$ to $(0, 1)$.*

1, 2, 4 $\Rightarrow$ There is no onto mapping from $\mathbb{Q}$ to $\mathbb{R}$.

$\because$ If $f : \mathbb{Q} \to \mathbb{R}$ was onto, then $g = f \circ f_2 \circ f_1 : \mathbb{N} \to \mathbb{R}$ would be onto, where $f_1 : \mathbb{N} \to \mathbb{Z}$ and $f_2 : \mathbb{Z} \to \mathbb{Q}$ are onto mappings.

Proof

1. There is an onto mapping from $\mathbb{N}$ to $\mathbb{Z}$:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots.$$

2. There is an onto mapping from $\mathbb{Z}$ to $\mathbb{Q}$:

$$\text{for } \mathbb{Z}_+ : \; 0, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \ldots,$$

$$\text{for } \mathbb{Z}_{--} : \; -\frac{1}{1}, -\frac{1}{2}, -\frac{2}{1}, -\frac{1}{3}, -\frac{2}{2}, -\frac{3}{1}, -\frac{1}{4}, -\frac{2}{3}, -\frac{3}{2}, -\frac{4}{1}, \ldots.$$

3. There is an onto mapping from $(0, 1)$ to $\mathbb{R}$:

$$f(x) = \tan\left(-\frac{\pi}{2} + \pi x\right).$$

Proof—Cantor's Diagonal Argument

4. There is no onto mapping from $\mathbb{N}$ to $(0,1)$:

   Assume that there were an onto mapping f:

$$
\begin{aligned}
1 &\mapsto 0.a_{11}a_{12}a_{13}\cdots \\
2 &\mapsto 0.a_{21}a_{22}a_{23}\cdots \\
&\vdots \\
n &\mapsto 0.a_{n1}a_{n2}a_{n3}\cdots a_{nn}\cdots \\
&\vdots
\end{aligned}
$$

   Let $x = 0.x_1x_2x_3\cdots x_n\cdots$ be defined by

$$
x_n = \begin{cases} 1 & \text{if } a_{nn} \text{ is even,} \\ 2 & \text{if } a_{nn} \text{ is odd.} \end{cases}
$$

   Then there is no $m \in \mathbb{N}$ such that $f(m) = x$, a contradiction.

Application: Lexicographic Preference Relation

- Let $\succsim$ be the lexicographic preference relation on $\mathbb{R}^2$, i.e., $(x, y) \succ (x', y')$ if and only if
  - $x > x'$ or
  - $x = x'$ and $y > y'$.

Proposition 1.11

There exists no utility function that represents the lexicographic preference relation $\succsim$.

Proof

Assume that $\succsim$ is represented by a utility function $u\colon \mathbb{R}^2 \to \mathbb{R}$.

- For each $x \in \mathbb{R}$, let

$$I_x = (\inf u(x, \mathbb{R}), \sup u(x, \mathbb{R})) \quad (\neq \emptyset),$$

  where $u(x, \mathbb{R}) = \{z \in \mathbb{R} \mid z = u(x, y) \text{ for some } y \in \mathbb{R}\} \neq \emptyset$.

- Note that $I_x \cap I_{x'} = \emptyset$ whenever $x \neq x'$.

- Define the function $f\colon \mathbb{Q} \to \mathbb{R}$ by

$$f(q) = \begin{cases} x & \text{if } q \in I_x, \\ 0 & \text{if there is no } x \in \mathbb{R} \text{ such that } q \in I_x. \end{cases}$$

- This f is onto, because for any $x \in \mathbb{R}$, there exists a $q \in \mathbb{Q}$ such that $q \in I_x$ by Proposition 1.3 (the denseness of $\mathbb{Q}$ in $\mathbb{R}$).

- But this contradicts Proposition 1.10.