7. Separating Hyperplane Theorems II

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Mathematics II

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Farkas' Lemma

Proposition 7.16 (Farkas' Lemma)

Let $A \in \mathbb{R}^{M \times N}$ and $b \in \mathbb{R}^N$.
The following conditions are equivalent:

1. *There exists $x \in \mathbb{R}^M$ such that $A^{\mathrm{T}}x = b$ and $x \geq 0$.*
2. *For any $y \in \mathbb{R}^N$, if $Ay \geq 0$, then $b^{\mathrm{T}}y \geq 0$.*

For proof, we will use the following:

Lemma 7.17

$\{A^{\mathrm{T}}x \in \mathbb{R}^N \mid x \in \mathbb{R}_+^M\}$ *is a closed set.*

Proof of Farkas' Lemma

- (1) $\Rightarrow$ (2): Immediate.
- (2) $\Rightarrow$ (1):

  Suppose that (1) does not hold.

  Let $K = \{A^{\mathrm{T}}x \in \mathbb{R}^N \mid x \in \mathbb{R}^M_+\}$. Then $b \notin K$.
- K is convex, and by Lemma 7.17 is closed.
- Then by the Separating Hyperplane Theorem, there exist $y \in \mathbb{R}^N$ with $y \neq 0$ and $c \in \mathbb{R}$ such that

  $$y^{\mathrm{T}}b < c \leq y^{\mathrm{T}}z \text{ for all } z \in K,$$

  and therefore, $y^{\mathrm{T}}b < \inf_{z\in K} y^{\mathrm{T}}z$.
- Since K is a cone, it follows that $\inf_{z\in K} y^{\mathrm{T}}z = 0$. ($\rightarrow$ Homework)
- Thus we have $y^{\mathrm{T}}b < 0$, and $y^{\mathrm{T}}A^{\mathrm{T}}x \geq 0$ for all $x \geq 0$, which implies that $y^{\mathrm{T}}A^{\mathrm{T}} \geq 0^{\mathrm{T}}$.

Proof of Lemma 7.17

Show that $K = \{A^{\mathrm{T}}x \in \mathbb{R}^N \mid x \in \mathbb{R}_+^M\}$ is closed.

- Denote the column vectors in A^{T} by $a^1, \ldots, a^M$, so that $K = \mathrm{Cone}\{a^1, \ldots, a^M\}$.
- Let $\{z^m\}$ be a sequence in K, and suppose that $z^m \to \bar{z}$.

  We want to show that $\bar{z} \in K$.
- By Carathéodory's Theorem, for each m, z^m is written as a conic combination of a linearly independent subset of $\{a^1, \ldots, a^M\}$.
- Since there are finitely many such subsets, there is a linearly independent subset $\{a^{i_1}, \ldots, a^{i_L}\}$ such that infinitely many elements of $\{z^m\}$ are written as its conic combinations.
- Denote $B = \begin{pmatrix} a^{i_1} & \cdots & a^{i_L} \end{pmatrix} \in \mathbb{R}^{N \times L}$, and denote the corresponding subsequence again by $\{z^m\}$.

- Denote $z^m = B\lambda^m$, where $\lambda^m \in \mathbb{R}^L_+$.
- We have $B^{\mathrm{T}} z^m = B^{\mathrm{T}} B\lambda^m$, where $B^{\mathrm{T}} B \in \mathbb{R}^{L\times L}$ is non-singular:
  - Let $B^{\mathrm{T}} Bx = 0$.
  - Then $x^{\mathrm{T}} B^{\mathrm{T}} Bx = 0$, where $x^{\mathrm{T}} B^{\mathrm{T}} Bx = \|Bx\|^2$.
  - Therefore, $x^{\mathrm{T}} B^{\mathrm{T}} Bx = 0$ if and only if $Bx = 0$.
  - Since the columns of B are linearly independent, this holds if and only if $x = 0$.
- Therefore, we have $\lambda^m = (B^{\mathrm{T}} B)^{-1} B^{\mathrm{T}} z^m$.
- By the continuity of $(B^{\mathrm{T}} B)^{-1} B^{\mathrm{T}} z$ in z, λ^m converges to $\bar{\lambda} = (B^{\mathrm{T}} B)^{-1} B^{\mathrm{T}} \bar{z}$, where $\bar{\lambda} \in \mathbb{R}^L_+$.
- Thus, by the continuity of $B\lambda$ in λ, we have $\bar{z} = \lim_{m\to\infty} B\lambda^m = B\bar{\lambda}$, so that $\bar{z} \in K$.

Variants of Farkas' Lemma

Proposition 7.18 (Farkas' Lemma: Inequality version)

Let $A \in \mathbb{R}^{M \times N}$ *and* $b \in \mathbb{R}^N$.
The following conditions are equivalent:

1. *There exists* $x \in \mathbb{R}^M$ *such that* $A^{\mathrm{T}} x \leq b$ *and* $x \geq 0$.
2. *For any* $y \in \mathbb{R}^N$, *if* $y \geq 0$ *and* $Ay \geq 0$, *then* $b^{\mathrm{T}} y \geq 0$.

Proof

- Condition (1) is equivalent to:

  There exist $x \in \mathbb{R}^M$ and $z \in \mathbb{R}^N$ such that $x \geq 0$, $z \geq 0$, and $A^{\mathrm{T}}x + z = b$,

  or $\begin{pmatrix} A^{\mathrm{T}} & I \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = b$.

- By Farkas' Lemma, this is equivalent to:

  For any $y \in \mathbb{R}^N$, if $\begin{pmatrix} A \\ I \end{pmatrix} y \geq 0$, then $b^{\mathrm{T}}y \geq 0$,

  or, if $y \geq 0$ and $Ay \geq 0$, then $b^{\mathrm{T}}y \geq 0$ (condition (2)).

Linear Programming

Let $A \in \mathbb{R}^{K \times N}$, $f \in \mathbb{R}^N$, $c \in \mathbb{R}^K$.

Primal problem:

$$
\text{(P)} \quad \begin{aligned} \max_{x \in \mathbb{R}^N} \quad & f^{\mathrm{T}} x \\ \text{s. t.} \quad & Ax \leq c \\ & x \geq 0. \end{aligned}
$$

Dual problem:

$$
\text{(D)} \quad \begin{aligned} \min_{\lambda \in \mathbb{R}^K} \quad & c^{\mathrm{T}} \lambda \\ \text{s. t.} \quad & A^{\mathrm{T}} \lambda \geq f \\ & \lambda \geq 0. \end{aligned}
$$

The Lagrangians for the two problems coincide
(the nonnegativity constraints aside):

$$
L(x, \lambda) = f^{\mathrm{T}} x - \lambda^{\mathrm{T}} (Ax - c) = c^{\mathrm{T}} \lambda - x^{\mathrm{T}} (A^{\mathrm{T}} \lambda - f).
$$

Weak Duality

Proposition 7.19

If $x \in \mathbb{R}^N$ *and* $\lambda \in \mathbb{R}^K$ *are feasible for* (P) *and* (D)*, respectively, then* $f^{\mathrm{T}} x \leq c^{\mathrm{T}} \lambda$*.*

Proof

- If $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^K$ are feasible for (P) and (D), then

$$f^{\mathrm{T}} x \leq (A^{\mathrm{T}} \lambda)^{\mathrm{T}} x = \lambda^{\mathrm{T}} (Ax) \leq \lambda^{\mathrm{T}} c.$$

Therefore, if $\bar{x} \in \mathbb{R}^N$ and $\bar{\lambda} \in \mathbb{R}^K$ are feasible and if $f^{\mathrm{T}} \bar{x} = c^{\mathrm{T}} \bar{\lambda}$, then $\bar{x}$ and $\bar{\lambda}$ are solutions to (P) and (D), respectively.

Strong Duality

Proposition 7.20

Suppose that both (P) *and* (D) *are feasible.*
Then both (P) *and* (D) *have solutions, and*

$$\max\{f^{\mathrm{T}}x \mid Ax \leq c,\ x \geq 0\} = \min\{c^{\mathrm{T}}\lambda \mid A^{\mathrm{T}}\lambda \geq f,\ \lambda \geq 0\}.$$

Proof

- Suppose that (P) and (D) are feasible.

  We want to show that there exist $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^K$ such that $Ax \leq c$, $A^{\mathrm{T}}\lambda \geq f$, $f^{\mathrm{T}}x \geq c^{\mathrm{T}}\lambda$, $x \geq 0$, and $\lambda \geq 0$, or

$$\begin{pmatrix} A & O \\ O & -A^{\mathrm{T}} \\ -f^{\mathrm{T}} & c^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} \leq \begin{pmatrix} c \\ -f \\ 0 \end{pmatrix}, \; x \geq 0, \; \lambda \geq 0.$$

- By Farkas' Lemma (inequality version; Proposition 7.18), this is equivalent to the condition that for all $p \in \mathbb{R}^K$, $q \in \mathbb{R}^N$, and $r \in \mathbb{R}$,

$$\begin{pmatrix} p^{\mathrm{T}} & q^{\mathrm{T}} & r \end{pmatrix} \begin{pmatrix} A & O \\ O & -A^{\mathrm{T}} \\ -f^{\mathrm{T}} & c^{\mathrm{T}} \end{pmatrix} \geq 0, \; p \geq 0, \; q \geq 0, \; r \geq 0$$

$$\Rightarrow \begin{pmatrix} p^{\mathrm{T}} & q^{\mathrm{T}} & r \end{pmatrix} \begin{pmatrix} c \\ -f \\ 0 \end{pmatrix} \geq 0.$$

- That is,

  (1) $A^{\mathrm{T}}p \geq rf,\ Aq \leq rc,\ p \geq 0,\ q \geq 0,\ r \geq 0$

  implies

  (2) $c^{\mathrm{T}}p - f^{\mathrm{T}}q \geq 0.$

  We want to show that this holds whenever (P) and (D) are feasible.

- For $r > 0$, (1) implies that q/r and p/r are feasible solutions to (P) and (D), so that we have $c^{\mathrm{T}}p - f^{\mathrm{T}}q = r[c^{\mathrm{T}}(p/r) - f^{\mathrm{T}}(q/r)] \geq 0$ by Weak Duality.

- For $r = 0$, let x and λ be feasible solutions to (P) and (D). From (1), we have

$$c^{\mathrm{T}}p - f^{\mathrm{T}}q \geq x^{\mathrm{T}}A^{\mathrm{T}}p - \lambda^{\mathrm{T}}Aq \geq 0.$$

Strong Duality

Proposition 7.21

1. *Suppose that* (D) *has a solution.*

   Then (P) *has a solution, and*

   $$\max\{f^{\mathrm{T}}x \mid Ax \leq c,\ x \geq 0\} = \min\{c^{\mathrm{T}}\lambda \mid A^{\mathrm{T}}\lambda \geq f,\ \lambda \geq 0\}.$$

2. *Suppose that* (P) *has a solution.*

   Then (D) *has a solution, and*

   $$\max\{f^{\mathrm{T}}x \mid Ax \leq c,\ x \geq 0\} = \min\{c^{\mathrm{T}}\lambda \mid A^{\mathrm{T}}\lambda \geq f,\ \lambda \geq 0\}.$$

Proof

- Suppose that (D) has a solution.

  In light of Proposition 7.20, it suffices to show that (P) has a feasible solution.

- Let $\lambda^* \in \mathbb{R}^K$ be a solution to (D).

  To apply Farkas' Lemma (Proposition 7.18), let $z \in \mathbb{R}^K$ be such that $A^{\mathrm{T}} z \geq 0$ and $z \geq 0$.

- Then $\lambda^* + z \geq 0$, and $A^{\mathrm{T}}(\lambda^* + z) = A^{\mathrm{T}}\lambda^* + A^{\mathrm{T}} z \geq f$, which means that $\lambda^* + z$ is feasible in (D).

- Therefore, by the optimality of λ^*, we have $0 \leq c^{\mathrm{T}}(\lambda^* + z) - c^{\mathrm{T}}\lambda^* = c^{\mathrm{T}} z$.

- By Proposition 7.18, there exists $x \in \mathbb{R}^N$ such that $Ax \leq c$ and $x \geq 0$.

Variants of Farkas' Lemma

Proposition 7.22 (Gale's Theorem)

Let $A \in \mathbb{R}^{M \times N}$ and $b \in \mathbb{R}^N$.
The following conditions are equivalent:

1. *There exists $x \in \mathbb{R}^M$ such that $A^{\mathrm{T}} x \le b$.*
2. *For any $y \in \mathbb{R}^N$, if $y \ge 0$ and $Ay = 0$, then $b^{\mathrm{T}} y \ge 0$.*

Proof

- Condition (1) is equivalent to:

  There exist $z^1 \in \mathbb{R}^M$ and $z^2 \in \mathbb{R}^M$ such that $z^1 \geq 0$, $z^2 \geq 0$, and $A^{\mathrm{T}}(z^1 - z^2) \leq b$,

  or $\begin{pmatrix} A^{\mathrm{T}} & -A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \leq b$.

- By Farkas' Lemma (inequality version; Proposition 7.18), this is equivalent to:

  For any $y \in \mathbb{R}^N$, if $y \geq 0$ and $\begin{pmatrix} A \\ -A \end{pmatrix} y \geq 0$, then $b^{\mathrm{T}} y \geq 0$,

  or, if $y \geq 0$ and $Ay = 0$, then $b^{\mathrm{T}} y \geq 0$ (condition (2)).

Variants of Farkas' Lemma

Proposition 7.23 (Gordan's Theorem)

Let $A \in \mathbb{R}^{M \times N}$*.*
The following conditions are equivalent:

1. *There exists* $x \in \mathbb{R}^M$ *such that* $A^{\mathrm{T}} x \gg 0$*.*
2. *For any* $y \in \mathbb{R}^N$*, if* $y \geq 0$ *and* $Ay = 0$*, then* $y = 0$*.*

Proof

- Condition (1) is equivalent to:

  There exists $x \in \mathbb{R}^M$ such that $-A^{\mathrm{T}}x \le -\mathbf{1}$.

- By Gale's Theorem (Proposition 7.22), this is equivalent to:

  For any $y \in \mathbb{R}^N$, if $y \ge 0$ and $(-A)y = 0$, then $(-\mathbf{1}^{\mathrm{T}})y \ge 0$,

  or $y \ge 0$ and $Ay = 0$, then $y = 0$ (condition (2)).

Variants of Farkas' Lemma

Proposition 7.24 (Ville/von Neumann-Morgenstern I)

Let $A \in \mathbb{R}^{M \times N}$.

The following conditions are equivalent:

1. *There exists* $x \in \mathbb{R}^M$ *such that* $A^{\mathrm{T}}x \gg 0$ *and* $x \gg 0$.
2. *For any* $y \in \mathbb{R}^N$, *if* $y \geq 0$ *and* $Ay \leq 0$, *then* $y = 0$.

Variants of Farkas' Lemma

- In fact,
  "there exists $x \in \mathbb{R}^M$ such that $A^{\mathrm{T}}x \gg 0$ and $x \gg 0$"
  is equivalent to
  "there exists $x \in \mathbb{R}^M$ such that $A^{\mathrm{T}}x \gg 0$ and $x \geq 0$".
  - Given an $x \geq 0$ in the latter, consider $x + \varepsilon\mathbf{1}$ for sufficiently small $\varepsilon > 0$.

Proposition 7.25 (Ville/von Neumann-Morgenstern II)

Let $A \in \mathbb{R}^{M \times N}$.
The following conditions are equivalent:

1. *There exists $x \in \mathbb{R}^M$ such that $A^{\mathrm{T}}x \gg 0$ and $x \geq 0$.*
2. *For any $y \in \mathbb{R}^N$, if $y \geq 0$ and $Ay \leq 0$, then $y = 0$.*

Proof of Proposition 7.24

- Condition (1) is equivalent to:

  There exists $x \in \mathbb{R}^M$ such that $\begin{pmatrix} A^{\mathrm{T}} \\ I \end{pmatrix} x \gg 0$.

- By Gordan's Theorem (Proposition 7.23), this is equivalent to:

  For any $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^M$,
  if $y \geq 0$, $z \geq 0$, and $\begin{pmatrix} A & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0$, then $\begin{pmatrix} y \\ z \end{pmatrix} = 0$.

- This is equivalent to:

  For any $y \in \mathbb{R}^N$,
  if $y \geq 0$ and $Ay \leq 0$, then $y = 0$ (condition (2)).

Variants of Farkas' Lemma

Proposition 7.26

Let $A \in \mathbb{R}^{M \times N}$.
The following conditions are equivalent:

1. *There exists $x \in \mathbb{R}^M$ such that $A^{\mathrm{T}} x \leq 0$ and $x \gg 0$.*
2. *For any $y \in \mathbb{R}^N$, if $y \geq 0$ and $Ay \geq 0$, then $Ay = 0$.*

Proof

- Condition (1) is equivalent to:

  There exists $x \in \mathbb{R}^M$ such that $\begin{pmatrix} A^{\mathrm{T}} \\ -I \end{pmatrix} x \leq \begin{pmatrix} 0 \\ -\mathbf{1} \end{pmatrix}$.

- By Gale's Theorem (Proposition 7.22), this is equivalent to:

  For any $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^M$,
  if $y \geq 0$, $z \geq 0$, and $\begin{pmatrix} A & -I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0$, then
  $\begin{pmatrix} 0 & -\mathbf{1}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq 0$.

- This is equivalent to:

  For any $y \in \mathbb{R}^N$,
  if $y \geq 0$ and $Ay \geq 0$, then $Ay = 0$ (condition (2)).

Variants of Farkas' Lemma

Proposition 7.27 (Stiemke's Lemma)

Let $A \in \mathbb{R}^{M \times N}$*.*

The following conditions are equivalent:

1. *There exists* $x \in \mathbb{R}^M$ *such that* $A^{\mathrm{T}} x = 0$ *and* $x \gg 0$*.*
2. *For any* $y \in \mathbb{R}^N$*, if* $Ay \geq 0$*, then* $Ay = 0$*.*

Proof

- Condition (1) is equivalent to:

  There exists $x \in \mathbb{R}^M$ such that $x \gg 0$ and $\begin{pmatrix} A^{\mathrm{T}} \\ -A^{\mathrm{T}} \end{pmatrix} x \leq 0$.

- By Proposition 7.26, this is equivalent to:

  For any $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^M$,
  if $y \geq 0$, $z \geq 0$, and $\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geq 0$, then
  $\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0$.

- This is equivalent to:

  For any $y \in \mathbb{R}^N$, if $Ay \geq 0$, then $Ay = 0$ (condition (2)).

Variants of Farkas' Lemma

Proposition 7.28 (Motzkin's Theorem)

Let $B \in \mathbb{R}^{M \times N}$, $C \in \mathbb{R}^{M \times K}$, $D \in \mathbb{R}^{M \times L}$.
The following conditions are equivalent:

1. *There exists no* $x \in \mathbb{R}^M$ *such that* $B^{\mathrm{T}} x \gg 0$, $C^{\mathrm{T}} x \geq 0$, *and* $D^{\mathrm{T}} x = 0$.
2. *There exist* $y_1 \in \mathbb{R}^N$, $y_2 \in \mathbb{R}^K$, *and* $y_3 \in \mathbb{R}^L$ *such that* $By_1 + Cy_2 + Dy_3 = 0$, $y_1 \geq 0$, $y_1 \neq 0$, *and* $y_2 \geq 0$.

- Proved using Farkas' Lemma.
- Proposition 7.23 (Gordan's Theorem), Propositions 7.24-7.25 (Ville's Theorem), Proposition 7.26, and Proposition 7.27 (Stiemke's Lemma) are all special cases of this theorem.

Efficient Production under Linear Technology

- For the production set $Y \subset \mathbb{R}^N$, $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Proposition 7.29

Let $Y = \{y \in \mathbb{R}^N \mid Ay \leq b\}$ *for some* $A \in \mathbb{R}^{M \times N}$ *and* $b \in \mathbb{R}^M$*. Then* $\bar{y} \in Y$ *is efficient if and only if there exists* $p \gg 0$ *such that*

$$p \cdot \bar{y} \geq p \cdot y \text{ for all } y \in Y.$$

- The "if" part holds for general set Y.

Proof

- The "if" part:

  If $\bar{y}$ is not efficient, i.e., $y' - \bar{y} \geq 0, \neq 0$ for some $y' \in Y$, then for any $p \gg 0$, we have $(y' - \bar{y})p > 0$ or $y'p > y^*p$.

- The "only if" part:

  Suppose that $\bar{y} \in Y$ is efficient.

- Write $A = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$ and $b = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}$ such that

  $$A^1 \bar{y} = b^1, \qquad A^2 \bar{y} \ll b^2,$$

  where $A^k \in \mathbb{R}^{M_k \times N}$, $b^k \in \mathbb{R}^{M_k}$, $k = 1, 2$, and $M_1 + M_2 = M$.

- By the efficiency of $\bar{y}$, $M_1 \geq 1$.

- By the efficiency of $\bar{y}$, there exists no $z \in \mathbb{R}^N$ such that $A^1 z \leq 0$, $z \geq 0$, $z \neq 0$.

  If there exists such z, then $A(\bar{y} + \varepsilon z) \leq b$ for sufficiently small $\varepsilon > 0$, where $\bar{y} + \varepsilon z \gneq \bar{y}$.

- By Proposition 7.25 (Ville's Theorem), there exists $x \in \mathbb{R}^{M_1}$ such that $(A^1)^{\mathrm{T}} x \gg 0$ and $x \geq 0$.

  Let $p = (A^1)^{\mathrm{T}} x$ ($\gg 0$).

- Then for any $y \in Y$ (where $A^1 y \leq b^1$), we have

$$p \cdot \bar{y} = x \cdot A^1 \bar{y} = x \cdot b^1,$$
$$p \cdot y = x \cdot A^1 y \leq x \cdot b^1,$$

  as desired.

Strict Dominance and Never Best Response

Consider a two-player normal form game:

- $S_1 = \{1, \ldots, M\}$: set of pure strategies of player 1 ($M \geq 2$)

  $S_2 = \{1, \ldots, N\}$: set of pure strategies of player 2 ($N \geq 2$)

- $\Delta(S_1) = \{x \in \mathbb{R}_+^M \mid x_1 + \ldots + x_M = 1\}$:
  set of mixed strategies of player 1

  $\Delta(S_2) = \{y \in \mathbb{R}_+^N \mid y_1 + \ldots + y_N = 1\}$:
  set of mixed strategies of player 2

- From player 1's point of view, $\Delta(S_2)$ is interpreted as the set of 1's *beliefs* over 2's strategies.

- Pure strategy $m \in S_1$ is identified with $e_m \in \Delta(S_1)$, the mth unit vector of $\mathbb{R}^M$.

- Payoff matrix for player 1:

  $$U = \begin{pmatrix} u_{11} & \cdots & u_{1N} \\ \vdots & \ddots & \vdots \\ u_{M1} & \cdots & u_{MN} \end{pmatrix} \in \mathbb{R}^{M \times N}$$

  (We only consider the incentives of player 1.)

- $e_m^{\mathrm{T}} U y \cdots$ payoff from $m \in S_1$ against $y \in \Delta(S_2)$
- $x^{\mathrm{T}} U y \cdots$ payoff from $x \in \Delta(S_1)$ against $y \in \Delta(S_2)$

- $m \in S_1$ is a best response to $y \in \Delta(S_2)$ if $e_m^{\mathrm{T}} U y \geq e_\ell^{\mathrm{T}} U y$ for all $\ell \in S_1$.
- $m \in S_1$ is a *never best response* if it is not a best response to any $y \in \Delta(S_2)$.
- $m \in S_1$ is *strictly dominated* if there exists $x \in \Delta(S_1)$ such that $e_m^{\mathrm{T}} U e_n < x^{\mathrm{T}} U e_n$ for all $n \in S_2$.

Proposition 7.30

In a two-player normal form game, $m \in S_1$ is a never best response if and only if it is strictly dominated.

- The result extends straightforwardly to (finite) games with more than two players if best response is defined with respect to *correlated* beliefs over opponents' strategies.

Proof

- Let
$$\tilde{U} = \begin{pmatrix} u_{11} - u_{m1} & \cdots & u_{1N} - u_{mN} \\ \vdots & \ddots & \vdots \\ u_{M1} - u_{m1} & \cdots & u_{MN} - u_{mN} \end{pmatrix}.$$
- $m \in S_1$ is a never best response
$\iff$ there exists no $y \geq 0$, $y \neq 0$, such that $\tilde{U}y \leq 0$
$\iff$ if $y \geq 0$ and $\tilde{U}y \leq 0$, then $y = 0$.
- $m \in S_1$ is strictly dominated
$\iff$ there exists $x \geq 0$, $x \neq 0$, such that $x^{\mathrm{T}}\tilde{U} \gg 0$.
- By Ville's Theorem (Proposition 7.25), these are equivalent.

Weak Dominance and Never Best Response

- $m \in S_1$ is *weakly dominated* if there exists $x \in \Delta(S_1)$ such that
  - $e_m^{\mathrm{T}} U e_n \leq x^{\mathrm{T}} U e_n$ for all $n \in S_2$, and
  - $e_m^{\mathrm{T}} U e_n < x^{\mathrm{T}} U e_n$ for some $n \in S_2$.

Proposition 7.31

In a two-player normal form game, $m \in S_1$ is a best response to some totally mixed strategy $y \in \Delta(S_2)$ if and only if it is not weakly dominated.

Proof

- Again let

$$\tilde{U} = \begin{pmatrix} u_{11} - u_{m1} & \cdots & u_{1N} - u_{mN} \\ \vdots & \ddots & \vdots \\ u_{M1} - u_{m1} & \cdots & u_{MN} - u_{mN} \end{pmatrix}.$$

- $m \in S_1$ is a best response to some totally mixed strategy
  $\iff$ there exists $y \gg 0$ such that $\tilde{U} y \leq 0$.
- $m \in S_1$ is not weakly dominated
  $\iff$ there exists no $x \geq 0$, $x \neq 0$, such that $x^{\mathrm{T}} \tilde{U} \gneq 0$
  $\iff$ if $x \geq 0$ and $x^{\mathrm{T}} \tilde{U} \geq 0$, then $x^{\mathrm{T}} \tilde{U} = 0$.
- By Proposition 7.26, these are equivalent.