On the Differentiability of the Support Function:

Mathematical Notes for Advanced Microeconomics

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1 Introduction

Given a nonempty set $K \subset \mathbb{R}^L$, the *concave support function* and the *convex support function* of K are defined respectively by

$$\mu_K(p) = \inf_{x \in K} p \cdot x,$$
$$\nu_K(p) = \sup_{x \in K} p \cdot x.$$

For a given utility function u, the expenditure function, as a function of price vector with a required utility level u fixed, is the concave support function of the "at-least-as-good-as set" $\{x \mid u(x) \geq u\}$, while for a production set Y, the profit function is the convex support function of Y. Here we think of the expenditure function as a primary example and thus refer to a concave support function simply as a support function (as in MWG), whereas in convex analysis textbooks, a support function usually refers to a convex support function.

We will allow for functions that may take values $-\infty$ and ∞.[1] A function $f \colon \mathbb{R}^L \to [-\infty, \infty]$ is concave if $\{(x, w) \in \mathbb{R}^L \times \mathbb{R} \mid w \leq f(x)\}$, which is called the *hypograph* of f and denoted $\operatorname{hyp} f$, is convex in $\mathbb{R}^L \times \mathbb{R}$. A function $f \colon \mathbb{R}^L \to [-\infty, \infty]$ is convex if $-f$ is concave. Equivalently, $f \colon \mathbb{R}^L \to [-\infty, \infty]$ is convex if the *epigraph* of f, $\operatorname{epi} f = \{(x, w) \in \mathbb{R}^L \times \mathbb{R} \mid w \geq f(x)\}$, is convex in $\mathbb{R}^L \times \mathbb{R}$. Note that a concave function defined on a subset X of $\mathbb{R}^L$ can be extended to a concave function defined on $\mathbb{R}^L$ by assigning the value $-\infty$ outside X.

In this document, we give a proof of the following theorem (MWG, Proposition 3.F.1):

[1] But we will mostly deal with concave functions f such that $f(x) < \infty$ for all x and $f(x) > -\infty$ for some x. Such a function is called a *proper* concave function, whereas we will not use this terminology, but refer as "$f \colon \mathbb{R}^L \to [-\infty, \infty)$ with $f \not\equiv -\infty$".

Theorem 1.1. *Let $K \subset \mathbb{R}^L$ be a nonempty closed convex set, and μ_K its support function, i.e., $\mu_K(p) = \inf_{x \in K} p \cdot x$. Let $\bar{p} \in \mathbb{R}^L$ be such that $\mu_K(\bar{p}) > -\infty$. Then there is a unique $\bar{x} \in K$ such that $\mu_K(\bar{p}) = \bar{p} \cdot \bar{x}$ if and only if μ_K is differentiable at $\bar{p}$. Moreover, in this case, $\nabla \mu_K(\bar{p}) = \bar{x}$.*

This is Corollary 25.1.3 in Rockafellar (1970), which shows up in Chapter 25 on page 243. Its proof can thus be obtained if one goes through all those 25 chapters in this book, but it is obviously a tough task. In these notes, we present a minimum amount of theorems from Rockafellar that are necessary to prove the above theorem.

2 Separating Hyperplane Theorem

The starting point is the following Separating Hyperplane Theorem as presented in Theorem M.G.2 in MWG.

Theorem 2.1. *Let $A \subset \mathbb{R}^N$ be a closed convex set, and let $\bar{x} \notin A$. Then there exist $p \in \mathbb{R}^N$ with $p \neq 0$ and $\alpha \in \mathbb{R}$ such that $p \cdot \bar{x} < \alpha \leq p \cdot x$ for all $x \in A$.*

The following is an alternative presentation of the above theorem involving the support function of the convex set in consideration.

Theorem 2.2. *Let $K \subset \mathbb{R}^L$ be a nonempty closed convex set, and μ_K its support function. Then,*

$$K = \{x \in \mathbb{R}^L \mid p \cdot x \geq \mu_K(p) \textit{ for all } p \in \mathbb{R}^L\}.$$

Proof. (i) $K \subset$ RHS: By the definition of μ_K.

(ii) $K \supset$ RHS: Suppose that $\bar{x} \notin K$. Then by Theorem 2.1, there exist $\bar{p} \in \mathbb{R}^L$ with $\bar{p} \neq 0$ and $\alpha \in \mathbb{R}$ such that $\bar{p} \cdot \bar{x} < \alpha \leq \bar{p} \cdot x$ for all $x \in K$, and hence $\bar{p} \cdot \bar{x} < \mu_K(\bar{p})$, which means that $\bar{x} \notin$ RHS. ∎

The following is a separation theorem for the hypograph of a concave function.

Theorem 2.3 (Theorem 12.1 p.102)**.** *Let $f \colon \mathbb{R}^L \to [-\infty, \infty)$ be a concave function, and let $(\bar{x}, \bar{w}) \notin \mathrm{cl}(\mathrm{hyp}\, f)$. Then there exist $p \in \mathbb{R}^L$ with $p \neq 0$ and $\alpha \in \mathbb{R}$ such that*

$$p \cdot \bar{x} - \bar{w} < \alpha \leq p \cdot x - w \textit{ for all } (x, w) \in \mathrm{cl}(\mathrm{hyp}\, f). \tag{2.1}$$

Proof. Assume that f is not identically $-\infty$ (otherwise the assertion holds trivially). Let $(\bar{x}, \bar{w}) \notin \mathrm{cl}(\mathrm{hyp}\, f)$. Since the closed set $\mathrm{cl}(\mathrm{hyp}\, f)$ is convex by Lemma A.2, it follows from Theorem 2.1 that there exist $(q, \gamma) \in \mathbb{R}^L \times \mathbb{R}$ and $\beta \in \mathbb{R}$ such that

$$q \cdot \bar{x} + \gamma \bar{w} < \beta \leq q \cdot x + \gamma w \text{ for all } (x, w) \in \mathrm{cl}(\mathrm{hyp}\, f). \tag{2.2}$$

There are two cases: (i) $\gamma \neq 0$ and (ii) $\gamma = 0$.

(i) $\gamma \neq 0$: Since in (2.2), w can be arbitrarily small, we must have $\gamma < 0$. Thus, we obtain (2.1) by dividing (2.2) by $-\gamma > 0$ and letting $p = -q/\gamma$ and $\alpha = -\beta/\gamma$.

(ii) $\gamma = 0$: From (2.2) we have

$$q \cdot \bar{x} - \beta < 0, \text{ and} \tag{2.3}$$

$$0 \leq q \cdot x - \beta \text{ for all } x \in \mathbb{R}^L \text{ such that } (x, w) \in \mathrm{cl}(\mathrm{hyp}\, f) \text{ for some } w \in \mathbb{R}. \tag{2.4}$$

We claim that there exist $p^0 \in \mathbb{R}^L$ with $p^0 \neq 0$ and $\alpha^0 \in \mathbb{R}$ such that

$$w \leq p^0 \cdot x - \alpha^0 \text{ for all } (x, w) \in \mathrm{cl}(\mathrm{hyp}\, f), \tag{2.5}$$

which can be constructed as follows: Take any $(x^0, w^0) \notin \mathrm{cl}(\mathrm{hyp}\, f)$ with $f(x^0) > -\infty$. By Theorem 2.1, there exist $(q^0, \gamma^0) \in \mathbb{R}^L \times \mathbb{R}$ and $\beta^0 \in \mathbb{R}$ such that

$$q^0 \cdot x^0 + \gamma^0 w^0 < \beta^0 \leq q^0 \cdot x + \gamma^0 w \text{ for all } (x, w) \in \mathrm{cl}(\mathrm{hyp}\, f). \tag{2.6}$$

By the first inequality in (2.6) with $(x, w) = (x^0, f(x^0))$, so we have $\gamma^0 w^0 < \gamma^0 f(x^0)$, which together with $w^0 > f(x^0)$ implies that $\gamma^0 < 0$. Then divide the second inequality in (2.6) by $-\gamma^0 > 0$ and let $p^0 = -q^0/\gamma^0$ and $\alpha^0 = -\beta^0/\gamma^0$.

By (2.4) and (2.5), for any $\lambda \geq 0$ we have $w \leq (p^0 \cdot x - \alpha^0) + \lambda(q \cdot x - \beta)$ for all $(x, w) \in \mathrm{cl}(\mathrm{hyp}\, f)$, or

$$\alpha^0 + \lambda\beta \leq (p^0 + \lambda q)x - w \text{ for all } (x, w) \in \mathrm{cl}(\mathrm{hyp}\, f).$$

By (2.3), for sufficiently large λ we also have $(p^0 \cdot \bar{x} - \alpha^0) + \lambda(q \cdot \bar{x} - \beta) < \bar{w}$, or

$$(p^0 + \lambda q)\bar{x} - \bar{w} < \alpha^0 + \lambda\beta.$$

Hence, with such a λ, we have the desired inequality (2.1) by setting $p = p^0 + \lambda q$ and $\alpha = \alpha^0 + \lambda\beta$. ▮

A concave function $f\colon \mathbb{R}^L \to [-\infty, \infty]$ is said to be *closed* if $\mathrm{hyp}\, f$ is closed in $\mathbb{R}^{L+1}$.

3 Properties of Support Functions

Proposition 3.1. *The support function μ_K of a nonempty set $K \subset \mathbb{R}^L$ is homogeneous of degree one, concave, and closed.*

Proof. For any $\alpha > 0$, we have $\mu_K(\alpha p) = \inf_{x \in K}(\alpha p) \cdot x = \alpha \inf_{x \in K} p \cdot x = \alpha \mu_K(p)$.

We have $\mathrm{hyp}\, \mu_K = \{(p, m) \mid m \leq \inf_{k \in K} p \cdot x\} = \{(p, m) \mid m \leq p \cdot x \text{ for all } x \in K\} = \bigcap_{x \in K}\{(p, m) \mid m \leq p \cdot x\}$, which is convex and closed since $\{(p, m) \mid m \leq p \cdot x\}$ is convex and closed for each $x \in K$. ▮

In fact, these are the properties that characterize support functions: we will show in Section 4 that any closed concave function that is homogeneous of degree one is the support function of some closed convex set.

The following proposition gives a characterization of the minimizers of the function $p \cdot x$ with respect to $x \in K$.

Proposition 3.2 (Corollary 23.5.3, p.219). *Let $K \subset \mathbb{R}^L$ be a nonempty closed convex set, and μ_K its support function. For $\bar{x} \in \mathbb{R}^L$ and $\bar{p} \in \mathbb{R}^L$, the following conditions are equivalent:*

(a) $\bar{x} \in K$ *and* $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$*;*

(b) $\mu_K(p) \leq \mu_K(\bar{p}) + (p - \bar{p}) \cdot \bar{x}$ *for all* $p \in \mathbb{R}^L$*.*

Proof. The condition (b) is equivalent to

$$\bar{p} \cdot \bar{x} - \mu_K(\bar{p}) \leq p \cdot \bar{x} - \mu_K(p) \text{ for all } p \in \mathbb{R}^L,$$

which in turn is equivalent to

$$\bar{p} \cdot \bar{x} - \mu_K(\bar{p}) \leq \inf_{p \in \mathbb{R}^L} p \cdot \bar{x} - \mu_K(p). \tag{3.1}$$

(i) (a) $\Rightarrow$ (b): Suppose that $\bar{x} \in K$ and $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$. Then we first have $\bar{p} \cdot \bar{x} - \mu_K(\bar{p}) = 0$. By the definition of μ_K, we also have $p \cdot \bar{x} - \mu_K(p) \geq 0$ for all $p \in \mathbb{R}^L$. Hence the condition (3.1) holds.

(ii) (b) $\Rightarrow$ (a): We first show that the condition (3.1) implies $\bar{x} \in K$. Suppose that $\bar{x} \notin K$. Then by Theorem 2.2, there exists $\bar{p} \in \mathbb{R}^L$ such that $\bar{p} \cdot \bar{x} < \mu_K(\bar{p})$. Denote $\bar{p} \cdot \bar{x} - \mu_K(\bar{p}) = -\gamma$, where $\gamma > 0$. Then, since μ_K is homogeneous of degree one (Proposition 3.1), for any $\alpha > 0$ we have $(\alpha\bar{p}) \cdot \bar{x} - \mu_K(\alpha\bar{p}) = \alpha(\bar{p} \cdot \bar{x} - \mu_K(\bar{p})) = -\alpha\gamma$, which implies that $\inf_{p \in \mathbb{R}^L} p \cdot \bar{x} - \mu_K(p) = -\infty$ in the right hand side of (3.1), while the left hand side is finite. Hence, (3.1) does not hold.

Given $\bar{x} \in K$, we have $\bar{p} \cdot \bar{x} \geq \mu_K(\bar{p})$ by the definition of μ_K, while, since $\inf_{p \in \mathbb{R}^L} p \cdot \bar{x} - \mu_K(p) \leq 0$ (let $p = 0$, then $0 \cdot \bar{x} - \mu_K(0) = 0$), we have $\bar{p} \cdot \bar{x} \leq \mu_K(\bar{p})$ by (3.1). Thus we have $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$. ∎

The function μ_K^* defined by

$$\mu_K^*(x) = \inf_{p \in \mathbb{R}^L} p \cdot x - \mu_K(p),$$

which shows up in the right hand side of (3.1), is called the *conjugate* of μ_K. The support function μ_K is actually the conjugate of the *indicator function* of K, the function ρ_K defined by

$$\rho_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ -\infty & \text{if } x \notin K, \end{cases}$$

since μ_K can be written as

$$\mu_K(p) = \inf_{x \in \mathbb{R}^L} p \cdot x - \rho_K(x).$$

Part (ii) in the proof of Proposition 3.2 in fact shows that, if K is a closed convex set, it holds that $\mu_K^* = \rho_K$ (where the Separating Hyperplane Theorem 2.1 and the homogeneity of μ_K are used to show that $\mu_K^* \leq \rho_K$). In the next section, we prove a more general result, Theorem 4.3, for conjugates of general concave functions, which contains this fact as a special case.

Remark 3.1. Proposition 3.2 is in fact the "subdifferential version" of Shephard's Lemma (MWG, Proposition 3.G.1; also known as McKenzie's Lemma). When condition (b) holds, $\bar{x}$ is called a *subgradient* of μ_K at $\bar{p}$. The set of all subgradients of μ_K at $\bar{p}$ is called the *subdifferential* of μ_K at $\bar{p}$ and is denoted by $\partial\mu_K(\bar{p})$. Proposition 3.2 says that if K is a nonempty closed convex set, then

$$\arg\min_{x \in K} \bar{p} \cdot x = \partial\mu_K(\bar{p}). \tag{3.2}$$

In the context of expenditure minimization, where $K = \{x \in \mathbb{R}^L_+ \mid u(x) \geq u\}$, the left hand side of (3.2) defines the Hicksian demand correspondence, while the right hand side is the subdifferential of the expenditure function.

4 Conjugates of Concave Functions

Definition 4.1. For a concave function $f \colon \mathbb{R}^L \to [-\infty, \infty)$ with $f \not\equiv -\infty$, the *conjugate* of f is the function $f^* \colon \mathbb{R}^L \to [-\infty, \infty)$ defined by

$$f^*(p) = \inf_{x \in \mathbb{R}^L} p \cdot x - f(x).$$

We write $(f^*)^* = f^{**}$.

Recall that a concave function is said to be closed if its hypograph is closed.

Proposition 4.1. *f^* is a closed concave function.*

Proof. By definition,

$$\begin{aligned}
\operatorname{hyp} f^* &= \{(p, m) \in \mathbb{R}^L \times \mathbb{R} \mid m \leq \inf\nolimits_{x \in \mathbb{R}^L} p \cdot x - f(x)\} \\
&= \{(p, m) \in \mathbb{R}^L \times \mathbb{R} \mid m \leq p \cdot x - f(x) \text{ for all } x \in \mathbb{R}^L\} \\
&= \bigcap_{x \in \mathbb{R}^L} \{(p, m) \in \mathbb{R}^L \times \mathbb{R} \mid m \leq p \cdot x - f(x)\}.
\end{aligned}$$

Since for each $x \in \mathbb{R}^L$, $\{(p, m) \in \mathbb{R}^L \times \mathbb{R} \mid m \leq p \cdot x - f(x)\}$ is convex and closed in $\mathbb{R}^L \times \mathbb{R}$, so is $\operatorname{hyp} f^*$. ▮

Proposition 4.2. *For a concave function f, if $f(x) > -\infty$ for some x, then $f^*(p) < \infty$ for all p, while if $f(x) < \infty$ for all x, then $f^*(p) > -\infty$ for some p.*

Proof. The latter implication follows from Theorem 2.3. ▮

The following Conjugate Duality Theorem is one of the fundamental results in convex analysis. It follows from Theorem 2.3.

Theorem 4.3 (Theorem 12.2, p.104)**.** *Let $f \colon \mathbb{R}^L \to [-\infty, \infty)$ be a concave function. Then* $\operatorname{hyp} f^{**} = \operatorname{cl}(\operatorname{hyp} f)$. *If f is closed, then $f^{**} = f$.*

Proof. If $f \equiv -\infty$, then $f^{**} \equiv -\infty$, so that the assertions hold. Assume thus that $f \not\equiv -\infty$.

(i) $\operatorname{hyp} f^{**} \supset \operatorname{cl}(\operatorname{hyp} f)$: Since $\operatorname{hyp} f^{**}$ is a closed set by Proposition 4.1, it is sufficient to show that $\operatorname{hyp} f^{**} \supset \operatorname{hyp} f$, i.e., $f^{**} \geq f$. Fix any $x \in \mathbb{R}^L$. Since by definition, $f^*(p) \leq p \cdot x - f(x)$ for all $p \in \mathbb{R}^L$, we have

$$\begin{aligned} f^{**}(x) &= \inf_{p \in \mathbb{R}^L} p \cdot x - f^*(p) \\ &\geq \inf_{p \in \mathbb{R}^L} p \cdot x - \bigl(p \cdot x - f(x)\bigr) = f(x). \end{aligned}$$

(ii) $\operatorname{hyp} f^{**} \subset \operatorname{cl}(\operatorname{hyp} f)$: Let $(\bar{x}, \bar{w}) \notin \operatorname{cl}(\operatorname{hyp} f)$. Then by Theorem 2.3, there exist $\bar{p} \in \mathbb{R}^L$ with $\bar{p} \neq 0$ and $\alpha \in \mathbb{R}$ such that

$$\bar{p} \cdot \bar{x} - \bar{w} < \alpha \leq \bar{p} \cdot x - w \text{ for all } (x, w) \in \operatorname{cl}(\operatorname{hyp} f).$$

In particular, we have

$$\bar{p} \cdot \bar{x} - \bar{w} < \alpha \leq \bar{p} \cdot x - f(x) \text{ for all } x \in \mathbb{R}^L \text{ such that } f(x) > -\infty.$$

Therefore, we have

$$\bar{p} \cdot \bar{x} - \bar{w} < \inf\{\bar{p} \cdot x - f(x) \mid f(x) > -\infty\} = f^*(\bar{p}),$$

and hence $\bar{w} > \bar{p} \cdot \bar{x} - f^*(\bar{p}) \geq f^{**}(\bar{x})$, which means that $(\bar{x}, \bar{w}) \notin \operatorname{hyp} f^{**}$. ∎

For a concave function $f \colon \mathbb{R}^L \to [-\infty, \infty)$, we define the closure of f to be the function whose hypograph equals the closure of $\operatorname{hyp} f$.[2]

Definition 4.2. For a concave function $f \colon \mathbb{R}^L \to [-\infty, \infty)$, the *closure* of f is the function g such that $\operatorname{hyp} g = \operatorname{cl}(\operatorname{hyp} f)$, and is denoted by $\operatorname{cl} f$.

By Theorem 4.3, $\operatorname{cl} f$ is well defined and equal to f^{**}. By definition, $\operatorname{cl} f \geq f$. f is closed if and only if $\operatorname{cl} f = f$.

The following is an application of Theorem 4.3, which shows that the closure of a homogeneous concave function can be expressed as the support function of some convex set.

Proposition 4.4 (Corollary 13.2.1, p.114). *If $e \colon \mathbb{R}^L \to [-\infty, \infty)$ with $e \not\equiv -\infty$ is concave and homogeneous of degree one, then*

$$(\operatorname{cl} e)(p) = \inf_{x \in V} p \cdot x$$

for all $p \in \mathbb{R}^L$, where

$$V = \{x \in \mathbb{R}^L \mid q \cdot x \geq e(q) \textit{ for all } q \in \mathbb{R}^L\}.$$

[2] The definition here is different from, but equivalent to that in Rockafellar.

Proof. By Theorem 4.3,

$$(\operatorname{cl} e)(p) = \inf_{x\in\mathbb{R}^L} p\cdot x - e^*(x), \tag{4.1}$$

where e^* is the conjugate of e, i.e.,

$$e^*(x) = \inf_{q\in\mathbb{R}^L} q\cdot x - e(q). \tag{4.2}$$

Note that $e^*(x) < \infty$ for all $x \in \mathbb{R}^L$, since by assumption, $e(p) > -\infty$ for some p. Since e is positively homogeneous of degree one, by (4.2) we have

$$\begin{aligned} e^*(x) &= \inf_{q\in\mathbb{R}^L} q\cdot x - \alpha e(\alpha^{-1}q) \\ &= \alpha\left[\inf_{q\in\mathbb{R}^L} (\alpha^{-1}q)\cdot x - e(\alpha^{-1}q)\right] \\ &= \alpha e^*(x) \end{aligned}$$

for $\alpha > 0$, which together with $e^*(x) < \infty$ implies

$$e^*(x) = \begin{cases} 0 & \text{if } e^*(x) \geq 0, \\ -\infty & \text{if } e^*(x) < 0. \end{cases}$$

Hence, by (4.1) we have

$$(\operatorname{cl} e)(p) = \inf_{x\colon e^*(x)\geq 0} p\cdot x = \inf_{x\in V} p\cdot x,$$

as claimed. ▮

Remark 4.1. This is Proposition 3.H.1 in MWG (that one can recover the "at-least-as-good-as sets" from the expenditure function) without differentiability. To accommodate the difference in the domain (the positivity of price vectors), extend the given function defined on $\mathbb{R}^L_{++}$ to $\mathbb{R}^L$ by assigning $-\infty$ outside $\mathbb{R}^L_{++}$, and then apply Lemma A.6. That is, given a function $e(\cdot,u)\colon \mathbb{R}^L_{++} \to [0,\infty)$ that is concave and homogeneous of degree one (with utility level u fixed), define $\bar{e}(\cdot,u)\colon \mathbb{R}^L \to [-\infty,\infty)$ by $\bar{e}(p,u) = e(p,u)$ for $p \in \mathbb{R}^L_{++}$ and $\bar{e}(p,u) = -\infty$ for $p \notin \mathbb{R}^L_{++}$. Clearly, $\bar{e}(\cdot,u)$ is still concave and homogeneous of degree one. Hence, by Proposition 4.4, we have $(\operatorname{cl}\bar{e})(p,u) = \inf_{x\in V_u} p\cdot x$, where $V_u = \{x \in \mathbb{R}^L \mid q\cdot x \geq \bar{e}(q,u) \text{ for all } q \in \mathbb{R}^L\} = \{x \in \mathbb{R}^L \mid q\cdot x \geq e(q,u) \text{ for all } q \in \mathbb{R}^L_{++}\}$. By Lemma A.6, $(\operatorname{cl}\bar{e})(p,u) = e(p,u)$ for all $p \in \mathbb{R}^L_{++}$.

5 Differentiability of Concave Functions

Let $f\colon \mathbb{R}^L \to [-\infty,\infty]$ be any function, and let $\bar{x} \in \mathbb{R}^L$ be such that $|f(\bar{x})| < \infty$. f is *differentiable* at $\bar{x}$ if there exists $\bar{p} \in \mathbb{R}^L$ such that

$$\lim_{\|z\|\to 0} \frac{f(\bar{x}+z) - f(\bar{x}) - \bar{p}\cdot z}{\|z\|} = 0,$$

i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $z \in \mathbb{R}^L$, $z \neq 0$,

$$\|z\| \leq \delta \Longrightarrow \frac{|f(\bar{x}+z) - f(\bar{x}) - \bar{p} \cdot z|}{\|z\|} \leq \varepsilon.$$

In this case, $\bar{p}$ equals $\nabla f(\bar{x}) = ((\partial f/\partial x_1)(\bar{x}), \ldots, (\partial f/\partial x_L)(\bar{x}))'$.

Lemma 5.1 (Theorem 23.1, p.213)**.** *Let $f\colon \mathbb{R}^L \to [-\infty, \infty]$ be a concave function, and let $\bar{x} \in \mathbb{R}^L$ be such that $|f(\bar{x})| < \infty$. For any $z \in \mathbb{R}^n$,*

$$\frac{f(\bar{x}+\lambda z) - f(\bar{x})}{\lambda} \qquad (\lambda > 0)$$

is a nonincreasing function of λ.

Proof. Given $\bar{x}$ such that $|f(\bar{x})| < \infty$, let

$$g(z) = f(\bar{x}+z) - f(\bar{x}),$$

which is concave by the concavity of f. Note that $g(0) = 0$.

Let $\lambda < \lambda'$ ($\lambda, \lambda' > 0$). We want to show that $g(\lambda z)/\lambda \geq g(\lambda' z)/\lambda'$. Assume that $g(\lambda' z) > -\infty$ (otherwise the inequality holds trivially). Take any $w' \leq g(\lambda' z)$. Since $(0,0), (\lambda' z, w') \in \operatorname{hyp} g$, we have

$$\left(1 - \frac{\lambda}{\lambda'}\right)(0,0) + \frac{\lambda}{\lambda'}(\lambda' z, w') = \left(\lambda z, \frac{\lambda}{\lambda'} w'\right) \in \operatorname{hyp} g,$$

and therefore,

$$g(\lambda z) \geq \frac{\lambda}{\lambda'} w'.$$

If $g(\lambda' z) < \infty$, then letting $w' = g(\lambda' z)$, we have $g(\lambda z) \geq (\lambda/\lambda') g(\lambda' z)$, hence $g(\lambda z)/\lambda \geq g(\lambda' z)/\lambda'$. If $g(\lambda' z) = \infty$, then w' can be arbitrarily large, so we have $g(\lambda z) = \infty$. ∎

For any function $f\colon \mathbb{R}^L \to [-\infty, \infty]$ and for $\bar{x} \in \mathbb{R}^L$ such that $|f(\bar{x})| < \infty$, if the limit

$$\lim_{\lambda \searrow 0} \frac{f(\bar{x}+\lambda z) - f(\bar{x})}{\lambda}$$

exists (in $[-\infty, \infty]$), then it is called the *one-sided directional derivative* of f at $\bar{x}$ with respect to z and is denoted by $f'(\bar{x}; z)$. Note that $f'(\bar{x}; 0) = 0$ by definition. If f is differentiable at $\bar{x}$, then, for each $z \in \mathbb{R}^L$, we have

$$f'(\bar{x}; z) = \nabla f(\bar{x}) \cdot z,$$

since, for $z \neq 0$,

$$\frac{f(\bar{x}+\lambda z) - f(\bar{x})}{\lambda} - \nabla f(\bar{x}) \cdot z = \frac{f(\bar{x}+\lambda z) - f(\bar{x}) - \nabla f(\bar{x}) \cdot z}{\|\lambda z\|} \|z\| \to 0$$

as $\lambda \searrow 0$. Lemma 5.1 implies that if f is a concave function, then, even f is not differentiable at $\bar{x}$, $f'(\bar{x}; z)$ exists, equal to

$$\sup_{\lambda > 0} \frac{f(\bar{x}+\lambda z) - f(\bar{x})}{\lambda}.$$

Lemma 5.2 (Theorem 23.1, p.213)**.** *Let $f\colon \mathbb{R}^L \to [-\infty,\infty]$ be a concave function, and let $\bar{x} \in \mathbb{R}^L$ be such that $|f(\bar{x})| < \infty$. The function $f'(\bar{x};\cdot)$ is homogeneous of degree one and concave.*

Proof. By definition, for any $\alpha > 0$,

$$\begin{aligned} f'(\bar{x};\alpha z) &= \lim_{\lambda \searrow 0} \frac{f(\bar{x}+\lambda(\alpha z)) - f(\bar{x})}{\lambda} \\ &= \alpha \lim_{\lambda \searrow 0} \frac{f(\bar{x}+(\lambda\alpha)z) - f(\bar{x})}{\alpha\lambda} = \alpha f'(\bar{x};z), \end{aligned}$$

which means that $f'(\bar{x};\cdot)$ is homogeneous of degree one.

For $\lambda > 0$, let

$$h_\lambda(z) = \frac{f(\bar{x}+\lambda z) - f(\bar{x})}{\lambda}.$$

which is concave in z by the concavity of f. Let $(z,w),(z',w') \in \operatorname{hyp} f'(\bar{x};\cdot)$. By the definition of $f'(\bar{x};\cdot)$, for any $\varepsilon > 0$ we have $w - \varepsilon \le h_\lambda(z)$ and $w' - \varepsilon \le h_\lambda(z')$ for some $\lambda > 0$. Then for any $\alpha \in [0,1]$, we have

$$(1-\alpha)w + \alpha w' - \varepsilon \le h_\lambda((1-\alpha)z + \alpha z') \le f'(\bar{x};(1-\alpha)z + \alpha z'),$$

where the first inequality follows from the concavity of h_λ, while the second inequality follows from Lemma 5.1. Since $\varepsilon > 0$ is arbitrary, this implies that $(1-\alpha)(w,z) + \alpha(w',z') \in \operatorname{hyp} f'(\bar{x};\cdot)$, i.e., $f'(\bar{x};\cdot)$ is a concave function. ∎

The following is the main result in these notes, from which Theorem 1.1 follows.

Theorem 5.3 (Theorem 25.1)**.** *Let $f\colon \mathbb{R}^L \to [-\infty,\infty]$ be a concave function, and let $\bar{x} \in \mathbb{R}^L$ be such that $|f(\bar{x})| < \infty$. If f is differentiable at $\bar{x}$, then $\nabla f(\bar{x})$ is the unique p that satisfies*

$$f(x) \le f(\bar{x}) + p\cdot(x-\bar{x}) \text{ for all } x \in \mathbb{R}^L. \tag{5.1}$$

Conversely, if there exists a unique p that satisfies (5.1)*, then f is differentiable at $\bar{x}$. In this case, the unique p equals $\nabla f(\bar{x})$.*

Proof. Suppose that f is differentiable at $\bar{x}$. The condition (5.1) is equivalent to the condition that

$$\frac{f(\bar{x}+\lambda z) - f(\bar{x})}{\lambda} \le p\cdot z \text{ for all } z \in \mathbb{R}^L \text{ and all } \lambda > 0$$

(let $x = \bar{x} + \lambda z$). Since by Lemma 5.1, the left hand side converges from below to $\nabla f(\bar{x})\cdot z$ as $\lambda \to 0$, this condition is equivalent to the condition that

$$\nabla f(\bar{x})\cdot z \le p\cdot z \text{ for all } z \in \mathbb{R}^L,$$

which holds if and only if $p = \nabla f(\bar{x})$.

Conversely, let p^* be the unique p that satisfies (5.1). Define the concave function g by

$$g(z) = f(\bar{x} + z) - f(\bar{x}) - p^* \cdot z,$$

where $g(0) = 0$, and by (5.1), $g(z) \leq 0$ for all $z \in \mathbb{R}^L$. Note that by the assumption that p^* is the unique p that satisfies (5.1), we have

$$g(z) \leq q \cdot z \text{ for all } z \in \mathbb{R}^L \iff q = 0. \tag{5.2}$$

We want to show that

$$\lim_{z \to 0} \frac{g(z)}{\|z\|} = 0,$$

i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \|z\| \leq \delta$, then $|g(z)|/\|z\| \leq \varepsilon$. In this case, $p^* = \nabla f(\bar{x})$.

For $\lambda > 0$, define the concave function h_λ by

$$h_\lambda(u) = \frac{g(\lambda u)}{\lambda},$$

where $h_\lambda(0) = 0$ and $h_\lambda(u) \leq 0$ for all $u \in \mathbb{R}^L$. Then define h by

$$h(u) = \lim_{\lambda \searrow 0} h_\lambda(u) \bigl(= g'(0; u)\bigr),$$

which is equal to $\sup_{\lambda > 0} h_\lambda(u)$ by Lemma 5.1. Note that $h \not\equiv -\infty$ (in fact, $h(0) = 0$) and $h < \infty$ (in fact, $h \leq 0$). By Lemma 5.2, h is concave and homogeneous of degree one. Hence, it follows from Proposition 4.4 that

$$(\operatorname{cl} h)(u) = \sup_{q \in V} q \cdot u$$

with

$$V = \{q \in \mathbb{R}^L \mid q \cdot v \geq h(v) \text{ for all } v \in \mathbb{R}^L\}.$$

Here, we have

$$\begin{aligned}
&q \cdot v \geq h(v) \text{ for all } v \in \mathbb{R}^L \\
&\iff q \cdot v \geq h_\lambda(v) \text{ for all } v \in \mathbb{R}^L \text{ and all } \lambda > 0 \\
&\iff q \cdot (\lambda v) \geq g(\lambda v) \text{ for all } v \in \mathbb{R}^L \text{ and all } \lambda > 0 \\
&\iff q \cdot z \geq g(z) \text{ for all } z \in \mathbb{R}^L \\
&\iff q = 0,
\end{aligned}$$

where the equivalence in the last line follows from (5.2). Therefore, we have $V = \{0\}$, which implies that $(\operatorname{cl} h)(u) = 0$ for all $u \in \mathbb{R}^L$. It follows from Lemma A.8 that

$h(u) = (\operatorname{cl} h)(u) = 0$ for all $u \in \mathbb{R}^L$. That is, for each $u \in \mathbb{R}^L$, we have $h_\lambda(u) \nearrow 0$ as $\lambda \searrow 0$.

Let B denote the unit ball in $\mathbb{R}^L$, and let $a^1, \ldots, a^m$ be any finite collection of points in $\mathbb{R}^L$ such that $B \subset \operatorname{co}\{a^1, \ldots, a^m\}$. Note that for each $\lambda > 0$ and for each $u = \sum_{i=1}^m \lambda_i a^i \in B$, we have

$$0 \geq h_\lambda(u) \geq \sum_{i=1}^m \lambda_i h_\lambda(a^i) \geq \min_{i=1,\ldots,m} h_\lambda(a^i)$$

by the concavity of h_λ. Fix any $\varepsilon > 0$, and for each $i = 1, \ldots, m$, let $\delta_i > 0$ be such that $h_\lambda(a^i) \geq -\varepsilon$ for all $\lambda \in (0, \delta_i]$. Set $\delta = \min_{i=1,\ldots,m} \delta_i > 0$. Now take any $z \neq 0$ such that $\|z\| \leq \delta$. Setting $\lambda = \|z\| \in (0, \delta]$ and $u = z/\|z\| \in B$, we have

$$0 \geq \frac{g(z)}{\|z\|} = h_\lambda(u) \geq \min_{i=1,\ldots,m} h_\lambda(a^i) \geq -\varepsilon,$$

where the last inequality follows from the choice of δ_i's. We thus have $|g(z)|/\|z\| \leq \varepsilon$. ▮

As already noted just after Proposition 3.2, when (5.1) holds, p is called a *subgradient* of f at $\bar{x}$, and the set of all subgradients of f at $\bar{x}$ is called the *subdifferential* of f at $\bar{x}$ and denoted by $\partial f(\bar{x})$ (and the correspondence $\partial f \colon x \mapsto \partial f(x)$ is called the subdifferential of f). In these terms, Theorem 5.3 reads: *Let $f \colon \mathbb{R}^L \to [-\infty, \infty]$ be a concave function, and let $\bar{x} \in \mathbb{R}^L$ be such that $|f(\bar{x})| < \infty$. Then f has a unique subgradient at $\bar{x}$ if and only if it is differentiable at $\bar{x}$, in which case $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$.*

We are now in a position to prove Theorem 1.1. It is an immediate consequence of Proposition 3.2 and Theorem 5.3.

Proof of Theorem 1.1. Denote $x^*(\bar{p}) = \arg\min_{x \in K} \bar{p} \cdot x$. Proposition 3.2 says that $x^*(\bar{p}) = \partial \mu_K(\bar{p})$. Theorem 5.3 thus implies that $x^*(\bar{p})$ is a singleton if and only if μ_K is differentiable at $\bar{p}$, and in this case, $x^*(\bar{p}) = \{\nabla \mu_K(\bar{p})\}$. ▮

Appendix: Topological Properties of Convex Sets

Lemma A.1 (Theorem 6.1, p.45)**.** *For any convex set $C \subset \mathbb{R}^N$, if $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$, then $(1-\lambda)x + \lambda y \in \operatorname{int} C$ for any $\lambda \in [0, 1)$.*

Proof. Let $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$. Fix any $\lambda \in [0, 1)$. We want to show that there exists $\bar{\varepsilon} > 0$ such that $(1-\lambda)x + \lambda y + \bar{\varepsilon} u \in C$ for all $u \in B$, where B is the unit ball in $\mathbb{R}^N$ around 0.

Since $x \in \operatorname{int} C$, we can take $\varepsilon_0 > 0$ such that $x + \varepsilon_0 u \in C$ for all $u \in B$. Given this $\varepsilon_0 > 0$, let $\bar{\varepsilon} = \varepsilon_0(1-\lambda)/(1+\lambda) > 0$. Since $y \in \operatorname{cl} C$, we can take $u_0 \in B$ such that $y - \bar{\varepsilon} u_0 \in C$.

Fix any $u \in B$. Let $u' = [\lambda/(1+\lambda)]u_0 + [1/(1+\lambda)]u$, where $u' \in B$ by the convexity of B. Then, $(1-\lambda)x + \lambda y + \bar{\varepsilon} u$ can be expressed as

$$(1-\lambda)x + \lambda y + \bar{\varepsilon} u = (1-\lambda)x + \lambda(y - \bar{\varepsilon} u_0) + \lambda \bar{\varepsilon} u_0 + \bar{\varepsilon} u$$

$$= (1-\lambda)\left(x + \frac{\lambda\bar{\varepsilon}}{1-\lambda}u_0 + \frac{\bar{\varepsilon}}{1-\lambda}u\right) + \lambda(y - \bar{\varepsilon}u_0)$$
$$= (1-\lambda)(x + \varepsilon_0 u') + \lambda(y - \bar{\varepsilon}u_0)$$

(note that $\bar{\varepsilon}\lambda/(1-\lambda) = \varepsilon_0\lambda/(1+\lambda)$, $\bar{\varepsilon}/(1-\lambda) = \varepsilon_0/(1+\lambda)$ by the definition of $\bar{\varepsilon}$). Since $x + \varepsilon_0 u' \in C$ by the choice of ε_0, we have $(1-\lambda)(x + \varepsilon_0 u') + \lambda(y - \bar{\varepsilon}u_0) \in C$ by the convexity of C. ∎

Lemma A.2 (Theorem 6.2, p.45). *For any convex set $C \subset \mathbb{R}^N$, $\operatorname{cl} C$ and $\operatorname{int} C$ are convex.*

Proof. By definition,

$$\begin{aligned} x \in \operatorname{cl} C &\iff \forall \varepsilon > 0\ \exists y \in C : \|y - x\| < \varepsilon \\ &\iff \forall \varepsilon > 0\ \exists y \in C\ \exists z \in B_\varepsilon : x = y + z, \end{aligned}$$

where $B_\varepsilon = \{z \in \mathbb{R}^N \mid \|z\| < \varepsilon\}$. Therefore, $\operatorname{cl} C = \bigcap_{\varepsilon>0}\{y + z \mid y \in C,\ z \in B_\varepsilon\}$. Since C and B_ε are convex, so is $C + B_\varepsilon = \{y + z \mid y \in A,\ z \in B_\varepsilon\}$, and hence $\operatorname{cl} A$, the intersection of a family of convex sets, is convex.

The convexity of $\operatorname{int} C$ follows from Lemma A.1. ∎

Lemma A.3 (Theorem 6.2, p.45). *For any convex set $C \subset \mathbb{R}^N$, if $\operatorname{int}(\operatorname{cl} C) \neq \emptyset$, then $\operatorname{int} C \neq \emptyset$.*

Proof. Suppose that $\operatorname{int}(\operatorname{cl} C) \neq \emptyset$ (which implies that $C \neq \emptyset$). Let $x^0 \in C$. We claim that there are $x^1, \ldots, x^N \in C$ such that $x^1 - x^0, \ldots, x^N - x^0$ are linearly independent. Indeed, suppose otherwise that $m < N$ is the largest number for which C contains some points $y^1, \ldots, y^m$ such that $y - x^0, \ldots, y^m - x^0$ are linearly independent. Let $M = \{x^0 + \sum_{i=1}^m \lambda_i(y^i - x^0) \mid \lambda_1, \ldots, \lambda_m \in \mathbb{R}\}$ (the affine space spanned by $x^0, y^1, \ldots, y^m$). Then, $C \subset M$ by the maximality of m, and since M is a closed set, $\operatorname{cl} C \subset M$. Since $\operatorname{int} M = \emptyset$, we have $\operatorname{int}(\operatorname{cl} C) = \emptyset$, contradicting our hypothesis.

Now let

$$S = \{x^0 + \textstyle\sum_{i=1}^N \lambda_i(x^i - x^0) \mid \lambda_i \geq 0,\ \sum_{i=1}^N \lambda_i \leq 1\}$$

(i.e., $S = \operatorname{co}\{x^0, x^1, \ldots, x^N\}$), where $S \subset C$ since C is convex. This set S has a nonempty interior: $\operatorname{int} S = \{x^0 + \sum_{i=1}^N \lambda_i(x^i - x^0) \mid \lambda_i > 0,\ \sum_{i=1}^N \lambda_i < 1\} \neq \emptyset$. Therefore we have $\operatorname{int} C \neq \emptyset$. ∎

Lemma A.4 (Theorem 6.3, p.46). *For any convex set $C \subset \mathbb{R}^N$, $\operatorname{int}(\operatorname{cl} C) = \operatorname{int} C$.*

Proof. First, since $C \subset \operatorname{cl} C$, we have $\operatorname{int} C \subset \operatorname{int}(\operatorname{cl} C)$.

We then show the converse inclusion. If $\operatorname{int}(\operatorname{cl} C) = \emptyset$, the conclusion holds trivially. Suppose that $\operatorname{int}(\operatorname{cl} C) \neq \emptyset$. Then, since C is convex, it follows from Lemma A.3 that

$\operatorname{int} C \neq \emptyset$. Let $z \in \operatorname{int}(\operatorname{cl} C)$. We want to show that $z \in \operatorname{int} C$. Take any $x \in \operatorname{int} C$ ($\neq \emptyset$). Suppose that $x \neq z$ (otherwise $z \in \operatorname{int} C$ holds trivially). For $\varepsilon > 0$, let

$$y = z - \varepsilon(x - z),$$

and let ε be sufficiently small so that $y \in \operatorname{int}(\operatorname{cl} C)$ and hence $y \in \operatorname{cl} C$. Then z can be written as $z = [1 - 1/(1+\varepsilon)]x + [1/(1+\varepsilon)]y$, where $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$. Therefore, by Lemma A.1 it follows that $z \in \operatorname{int} C$. ▮

Remark A.1. From this lemma, the statement in the proof of Theorem M.G.3 in MWG follows, that for a convex set B, if $x \notin \operatorname{int} B$, then "we can find a sequence $x^m \to x$ such that, for all m, x^m is not an element of the closure of set B", which sounds "intuitive", but is not obvious at all. In fact, Lemmas A.3 and A.4 do not hold if one drops the convexity. For example, let $N = 1$ and $C = \mathbb{Q} \cap [0, 1]$. Then $\operatorname{cl} C = [0, 1]$ and therefore $\operatorname{int}(\operatorname{cl} C) = (0, 1)$ and $\operatorname{bdry}(\operatorname{cl} C) = \{0, 1\}$, whereas $\operatorname{int} C = \emptyset$ and $\operatorname{bdry} C = [0, 1]$.

For a function $f\colon \mathbb{R}^L \to [-\infty, \infty)$, we define

$$\operatorname{dom} f = \{x \in \mathbb{R}^L \mid f(x) > -\infty\}.$$

It is called the *effective domain* of f. Note that $\operatorname{dom} f$ is a convex set if f is a concave function.

Lemma A.5 (Lemma 7.3, p.54)**.** *For any concave function* $f\colon \mathbb{R}^L \to [-\infty, \infty)$,

$$\operatorname{int}(\operatorname{hyp} f) = \{(x, w) \in \mathbb{R}^L \times \mathbb{R} \mid x \in \operatorname{int}(\operatorname{dom} f),\ w < f(x)\}.$$

Proof. First, if $(x, w) \in \operatorname{int}(\operatorname{hyp} f)$, then for sufficiently small $\varepsilon > 0$, we have $(x + \varepsilon u, w + \varepsilon) \in \operatorname{hyp} f$ for any $u \in B$, where B is the unit ball in $\mathbb{R}^L$, which implies that $x + \varepsilon u \in \operatorname{dom} f$, and hence $x \in \operatorname{int}(\operatorname{dom} f)$, and $w + \varepsilon \leq f(x)$, and hence $w < f(x)$.

Second, take any $(\bar{x}, \bar{w})$ such that $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$ and $\bar{w} < f(\bar{x})$. For $\varepsilon > 0$, let $a^0 = \bar{x} - [\varepsilon/(n+1)]\mathbf{1}$ and $a^i = a^0 + \varepsilon e_i$, $i = 1, \ldots, L$, where $\mathbf{1} = (1, \ldots, 1)$ and e_i is the ith unit vector in $\mathbb{R}^L$, and let $\varepsilon > 0$ be sufficiently small so that $a^0, a^1, \ldots, a^L \in \operatorname{dom} f$. Let

$$P = \{a^0 + \textstyle\sum_{i=1}^L \lambda_i (a^i - a^0) \mid \lambda_1, \ldots, \lambda_L \geq 0,\ \textstyle\sum_{i=1}^L \lambda_i \leq 1\}$$

(i.e., $P = \operatorname{co}\{a^0, a^1, \ldots, a^L\}$). Note that $\bar{x} = [1/(n+1)] \sum_{i=0}^L a^i \in \operatorname{int} P = \{a^0 + \sum_{i=1}^L \lambda_i (a^i - a^0) \mid \lambda_1, \ldots, \lambda_L > 0,\ \sum_{i=1}^L \lambda_i < 1\}$. Let

$$\alpha = \min\{f(a^0), f(a^1), \ldots, f(a^L)\}.$$

Then for any $x \in P$, which can be written as $x = \sum_{i=0}^L \lambda_i a^i$ with $\lambda_0, \lambda_1, \ldots, \lambda_L \geq 0$ and $\sum_{i=0}^L \lambda_i = 1$, we have

$$f(x) \geq \sum_{i=0}^{L} \lambda_i f(a^i) \geq \left(\sum_{i=0}^{L} \lambda_i\right) \alpha = \alpha$$

by the concavity of f. Hence the open set $\operatorname{int} P \times (-\infty, \alpha)$ is contained in $\operatorname{hyp} f$.

If $\bar{w} < \alpha$, then $(\bar{x}, \bar{w}) \in \operatorname{int} P \times (-\infty, \alpha) \subset \operatorname{hyp} f$, and hence $(\bar{x}, \bar{w}) \in \operatorname{int}(\operatorname{hyp} f)$. If $\bar{w} \geq \alpha$, take any $w^0 < \alpha$, so that $(\bar{x}, w^0) \in \operatorname{int}(\operatorname{hyp} f)$. Then write $(\bar{x}, \bar{w}) = (1 - \lambda)(\bar{x}, w^0) + \lambda(\bar{x}, f(\bar{x}))$ with $\lambda = (\bar{w} - w^0)/(f(\bar{x}) - w^0) \in [0, 1)$. Since $(\bar{x}, w^0) \in \operatorname{int}(\operatorname{hyp} f)$ and $(\bar{x}, f(\bar{x})) \in \operatorname{hyp} f$, and $\operatorname{hyp} f$ is convex, we have $(\bar{x}, \bar{w}) \in \operatorname{int}(\operatorname{hyp} f)$ by Lemma A.1. ∎

Recall from Definition 4.2 that for a concave function $f \colon \mathbb{R}^L \to [-\infty, \infty)$, the closure of f, $\operatorname{cl} f$, is the function such that $\operatorname{hyp}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{hyp} f)$.

Lemma A.6 (Theorem 7.4, p.56)**.** *For any concave function $f \colon \mathbb{R}^L \to [-\infty, \infty)$, $(\operatorname{cl} f)(x) = f(x)$ for all $x \in \operatorname{int}(\operatorname{dom} f)$.*

Proof. Let $x \in \operatorname{int}(\operatorname{dom} f)$. By definition, $f(x) \leq (\operatorname{cl} f)(x)$. To show the converse inequality, take any $w \in \mathbb{R}$ such that $w \leq (\operatorname{cl} f)(x)$. We want to show that $w \leq f(x)$. Take a $w' \in \mathbb{R}$ such that $w' < f(x)$. Then, $(x, w') \in \operatorname{int}(\operatorname{hyp} f)$ by Lemma A.5. Since $(x, w) \in \operatorname{hyp}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{hyp} f)$, for any $\varepsilon \in (0, 1]$ we have $(1 - \varepsilon)(x, w) + \varepsilon(x, w') = (x, (1 - \varepsilon)w + \varepsilon w') \in \operatorname{int}(\operatorname{hyp} f)$ by Lemma A.1, and hence,

$$(1 - \varepsilon)w + \varepsilon w' < f(x)$$

again by Lemma A.5. Since this holds for any $\varepsilon > 0$, we have $w \leq f(x)$. ∎

Remark A.2. Lemma A.6 implies in particular that any concave function f such that $|f| < \infty$ is closed.

Lemma A.7 (Theorem 7.4, p.56)**.** *For any concave function $f \colon \mathbb{R}^L \to [-\infty, \infty)$, $\operatorname{int}(\operatorname{dom}(\operatorname{cl} f)) = \operatorname{int}(\operatorname{dom} f)$.*

Proof. We first show that $\operatorname{dom}(\operatorname{cl} f) \subset \operatorname{cl}(\operatorname{dom} f)$. Since $\operatorname{hyp} f \subset \operatorname{dom} f \times \mathbb{R}$, we have $\operatorname{cl}(\operatorname{hyp} f) \subset \operatorname{cl}(\operatorname{dom} f) \times \mathbb{R}$. Since $\operatorname{cl}(\operatorname{hyp} f) = \operatorname{hyp}(\operatorname{cl} f)$ by definition, we have $\operatorname{hyp}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{dom} f) \times \mathbb{R}$, and therefore $\operatorname{dom}(\operatorname{cl} f) \subset \operatorname{cl}(\operatorname{dom} f)$.

By this and Lemma A.4, we have $\operatorname{int}(\operatorname{dom}(\operatorname{cl} f)) \subset \operatorname{int}(\operatorname{cl}(\operatorname{dom} f)) = \operatorname{int}(\operatorname{dom} f)$. The converse inclusion holds by definition. ∎

Corollary A.8. *For any concave function $f \colon \mathbb{R}^L \to [-\infty, \infty)$, if $(\operatorname{cl} f)(x) > -\infty$ for all $x \in \mathbb{R}^L$, then $\operatorname{cl} f = f$.*

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