On the Differentiability of the Value Function:

Supplementary Notes for Mathematics for Economists

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In this document, by applying the argument of Milgrom and Segal (2002, Corollary 3), we present a proof of the differentiability of the value function for a dynamic optimization problem under the standard concavity assumption, that the return function is concave in both the choice variable and the state variable, as well as a version of an interiority assumption, that the optimal solution remains in the choice set for a neighborhood of the given parameter value. As demonstrated by Milgrom and Segal (2002, Corollary 3), these assumptions allow an elementary proof, where we do not need the fancy theorem by Rockafellar (1970, Theorem 25.1) referred to by Benveniste and Scheinkman (1979) (and Stokey and Lucas (1989)), which involves a separation theorem. Our interiority assumption (condition (b) in Theorem 2 or Corollary 3 below) is what is actually used in the proof of Stokey and Lucas (1989, Theorem 4.11), which is slightly weaker than that of Benveniste and Scheinkman (1979, Assumption 4).

Let $X \subset \mathbb{R}^\ell$ be a non-empty convex set, and $\Gamma\colon X \to X$ a non-empty valued correspondence whose graph

$$A = \{(x, y) \in X \times X \mid y \in \Gamma(x)\}$$

is convex. Let $F\colon A \to \mathbb{R}$ be the one-period return function. Our main assumption is that F is a concave function.

Given $x_0 \in X$, let

$$\Pi'(x_0) = \{\{x_t\}_{t=1}^\infty \mid x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, \ldots\}.$$

A typical element in $\Pi'(x_0)$ will be denoted by $\boldsymbol{x} = (x_1, x_2, \ldots)$. We assume that for all $x_0 \in X$ and $\boldsymbol{x} \in \Pi'(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^n \beta^t F(x_t, x_{t+1})$ exists in $\overline{\mathbb{R}}$.

Let $u\colon \mathcal{X} \to \overline{\mathbb{R}}$ be defined by

$$u(x_0, \boldsymbol{x}) = \sum_{t=0}^\infty \beta^t F(x_t, x_{t+1}),$$

where $\mathcal{X} = \{(x_0, \boldsymbol{x}) \mid x_0 \in X,\ \boldsymbol{x} \in \Pi'(x_0)\}$. The value function $v^*\colon X \to \overline{\mathbb{R}}$ is defined by

$$v^*(x_0) = \sup_{\boldsymbol{x} \in \Pi'(x_0)} u(x_0, \boldsymbol{x}). \tag{SP}$$

To simplify the arguments, we assume that for all $x_0 \in X$, $u(x_0, \boldsymbol{x}) > -\infty$ for some $\boldsymbol{x} \in \Pi'(x_0)$, so that $v^*(x_0) > -\infty$ for all $x_0 \in X$.

A function $f\colon X \to \overline{\mathbb{R}}$ is concave if its hypograph $\{(x, \mu) \in X \times \mathbb{R} \mid \mu \leq f(x)\}$ is a convex set in $\mathbb{R}^{\ell+1}$. Observe that f is concave if and only if $f((1-\lambda)x + \lambda x') > (1-\lambda)a + \lambda a'$ for all $\lambda \in (0,1)$ whenever $f(x) > a$ and $f(x') > a'$.

Proposition 1. *Assume that F is concave. Then v^* is concave.*

Proof. Suppose that $v^*(x_0) > a$ and $v^*(x'_0) > a'$. Then by definition, there exist $\boldsymbol{x} \in \Gamma(x_0)$ and $\boldsymbol{x}' \in \Gamma(x'_0)$ such that $u(x_0, \boldsymbol{x}) > a$ and $u(x'_0, \boldsymbol{x}') > a'$. Let $\lambda \in (0,1)$. Since F is a concave function, we have

$$\begin{aligned}\sum_{t=0}^{n} \beta^t F((1-\lambda)x_t + \lambda x'_t, (1-\lambda)x_{t+1} + \lambda x'_{t+1}) \\ \geq (1-\lambda)\sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \lambda \sum_{t=0}^{n} \beta^t F(x'_t, x'_{t+1})\end{aligned}$$

for all n, and therefore,

$$\begin{aligned}u((1-\lambda)x_0 + \lambda x'_0, (1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{x}') &\geq (1-\lambda)u(x_0, \boldsymbol{x}) + \lambda u(x'_0, \boldsymbol{x}') \\ &> (1-\lambda)a + \lambda a',\end{aligned}$$

which implies that $v^*((1-\lambda)x_0 + \lambda x'_0) > (1-\lambda)a + \lambda a'$. ▮

Note that, v^* being concave, if $v^*(x) < \infty$ for some $x \in \operatorname{int} X$, then $v^*(x) < \infty$ for all $x \in X$, since $v^*(x) > -\infty$ for all $x \in X$ by assumption.

For $i = 1, \dots, \ell$, we write F_i and v^*_i for the partial derivatives of F and v^* with respect to the ith argument, i.e.,

$$F_i(x, y) = \frac{\partial F}{\partial x_i}(x, y), \quad v^*_i(x) = \frac{\partial v^*}{\partial x_i}(x)$$

(when they exist).

Theorem 2. *Assume that F is concave. Suppose that*

(a) $x_0 \in \operatorname{int} X$, $v^*(x_0) < \infty$, $\boldsymbol{x}^* \in \arg\max_{\boldsymbol{x} \in \Pi'(x_0)} u(x_0, \boldsymbol{x})$,

(b) $x^*_1 \in \Gamma(x)$ *for all x in some neighborhood $D \subset X$ of x_0, and*

(c) $F_i(x_0, x^*_1)$ *exists.*

Then $v^*_i(x_0)$ exists, and $v^*_i(x_0) = F_i(x_0, x^*_1)$.

Proof. By Proposition 1, v^* is concave, and since $x_0 \in \operatorname{int} X$ and $v^*(x_0) < \infty$ by assumption, $v^*(x) < \infty$ for all $x \in X$. Let $\bar{\varepsilon} > 0$ be such that $\{(x_i, x_{0,-i}) \in X \mid x_i \in (x_{0,i} - \bar{\varepsilon}, x_{0,i} + \bar{\varepsilon})\} \subset D$. Take any $\varepsilon \in (0, \bar{\varepsilon}]$. By the concavity (and the finiteness) of v^*, we have $v^*(x_0) \geq (1/2)v^*(x_0 - \varepsilon e_i) + (1/2)v^*(x_0 + \varepsilon e_i)$, and therefore,

$$\frac{v^*(x_0) - v^*(x_0 - \varepsilon e_i)}{\varepsilon} \geq \frac{v^*(x_0 + \varepsilon e_i) - v^*(x_0)}{\varepsilon},$$

where $e_i \in \mathbb{R}^\ell$ is the ith unit vector. Since $\boldsymbol{x}^* \in \Pi'(x_0 - \varepsilon e_i)$ and $\boldsymbol{x}^* \in \Pi'(x_0 + \varepsilon e_i)$ by the choice of ε, we have

$$\begin{aligned} v^*(x_0) - v^*(x_0 - \varepsilon e_i) &\leq u(x_0, \boldsymbol{x}^*) - u(x_0 - \varepsilon e_i, \boldsymbol{x}^*) = F(x_0, x_1^*) - F(x_0 - \varepsilon e_i, x_1^*), \\ v^*(x_0 + \varepsilon e_i) - v^*(x_0) &\geq u(x_0 + \varepsilon e_i, \boldsymbol{x}^*) - u(x_0, \boldsymbol{x}^*) = F(x_0 + \varepsilon e_i, x_1^*) - F(x_0, x_1^*) \end{aligned}$$

by the definition of v^*. Hence, we have

$$\begin{aligned} \frac{F(x_0, x_1^*) - F(x_0 - \varepsilon e_i, x_1^*)}{\varepsilon} &\geq \frac{v^*(x_0) - v^*(x_0 - \varepsilon e_i)}{\varepsilon} \\ &\geq \frac{v^*(x_0 + \varepsilon e_i) - v^*(x_0)}{\varepsilon} \geq \frac{F(x_0 + \varepsilon e_i, x_1^*) - F(x_0, x_1^*)}{\varepsilon}. \end{aligned}$$

Now let $\varepsilon \to 0$. By the partial differentiability of F, the left most and the right most terms converge to $F_i(x_0, x_1^*)$, so that the two terms in between converge as well, and their limit equals $F_i(x_0, x_1^*)$. ∎

If $F(\cdot, x_1^*)$ is continuously differentiable at x_0, then v^* is continuously differentiable at x_0, and $\nabla v^*(x_0) = \nabla_x F(x_0, x_1^*)$.

Corollary 3. *Assume that F is concave. Suppose that v is a solution to*

$$v(x) = \sup_{y \in \Gamma(x)} F(x, y) + \beta v(y) \quad (x \in X) \tag{FE}$$

and satisfies $\lim_{t\to\infty} \beta^t v(x_t) = 0$ for all $x \in X$ and $\boldsymbol{x} \in \Pi'(x)$. Suppose that

(a) *$x_0 \in \operatorname{int} X$, $v(x_0) < \infty$, $y^* \in \arg\max_{y \in \Gamma(x_0)} F(x, y) + \beta v(y)$,*

(b) *$y^* \in \Gamma(x)$ for all x in some neighborhood $D \subset X$ of x_0, and*

(c) *$F_i(x_0, y^*)$ exists.*

Then $v_i(x_0)$ exists, and $v_i(x_0) = F_i(x_0, y^*)$.

The condition as stated in Stokey and Lucas (1989, Theorem 4.11) is in fact a sufficient condition for condition (b) under the assumption that Γ is convex valued and lower semi-continuous. Recall that the correspondence $\Gamma\colon X \to Y$ is lower semi-continuous if, for all $x \in X$, for any open set V such that $\Gamma(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $\Gamma(x') \cap V \neq \emptyset$ for all $x' \in U$.

Proposition 4. *Suppose that $\Gamma\colon X \to \mathbb{R}^\ell$ is a convex valued, lower semi-continuous correspondence. If $x_0 \in \operatorname{int} X$ and $y_0 \in \operatorname{int}\Gamma(x_0)$, then there exists a neighborhood D of x_0 such that $y_0 \in \Gamma(x)$ for all $x \in D$.*

Proof. Since $y_0 \in \operatorname{int}\Gamma(x_0)$ where $\Gamma(x_0)$ is a convex set, there exist $z_0, z_1, \dots, z_\ell \in \Gamma(x_0)$ such that $y_0 \in \operatorname{int}\operatorname{conv}\{z_0, z_1, \dots, z_\ell\}$. Let $V_0, V_1, \dots, V_\ell$ be open neighborhoods of $z_0, z_1, \dots, z_\ell$, respectively, such that $y_0 \in \operatorname{conv}\{z'_0, z'_1, \dots, z'_\ell\}$ for any $z'_0 \in V_0, z'_1 \in V_1, \dots, z'_\ell \in V_\ell$. Since Γ is lower semi-continuous, there exist neighborhoods $U_0, U_1, \dots, U_\ell$ of x_0 such that, for each $i = 0, 1, \dots, \ell$, $\Gamma(x) \cap V_i \neq \emptyset$ for any $x \in U_i$. Now let $D = \cap_{i=0}^{\ell} U_i$, which is a neighborhood of x_0. Consider any $x \in D$, and pick any $z'_i \in \Gamma(x) \cap V_i$ for each $i = 0, 1, \dots, \ell$. Then by construction, $y_0 \in \operatorname{conv}\{z'_0, z'_1, \dots, z'_\ell\}$. Since $\Gamma(x)$ is convex, we thus have $y_0 \in \Gamma(x)$. ∎

Note that for the proposition to hold, the domain X of Γ can be any topological space, while we exploited the finite dimensionality of the codomain.

The convexity of Γ is indispensable. To see this, let $\ell = 1$ and $X = \mathbb{R}$, and let Γ be defined by

$$\Gamma(x) = \{y \in \mathbb{R} \mid y \geq |x| \text{ or } y \leq -|x|\},$$

which is continuous, i.e., upper and lower semi-continuous (one can modify it to be compact valued). Then, $0 \in \operatorname{int}\Gamma(0)$ while $0 \notin \Gamma(x)$ for all $x \neq 0$.

Example 1. Let $\ell = 1$ and $X = \mathbb{R}_+$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable concave function, and $r\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ a continuously differentiable concave function that is non-decreasing in the second variable, and let $\Omega(x) = [0, f(x)]$ for $x \in X$. Consider

$$\begin{aligned} v^*(x_0) = \sup_{\{c_t\}_{t=0}^\infty} & \sum_{t=0}^{\infty} \beta^t r(x_t, c_t) \\ \text{s.t. } & c_t \in \Omega(x_t), \\ & x_{t+1} = f(x_t) - c_t, \quad x_0 \in X : \text{given}, \end{aligned} \tag{1}$$

where we assume that the infinite sum is always well defined. One can verify that v^* is a concave function. For simplicity, we assume that $|v^*(x_0)| < \infty$ for all $x_0 \in X$. The value function v^* satisfies the Bellman equation:

$$v^*(x) = \sup_{c \in \Omega(x)} r(x, c) + \beta v^*(f(x) - c) \qquad (x \in X). \tag{2}$$

Let the correspondence $\Gamma\colon X \to X$ be defined by

$$\Gamma(x) = \{y \in X \mid y = f(x) - c \text{ for some } c \in \Omega(x)\} = [0, f(x)],$$

which is continuous and has a convex graph, and the function $F\colon A \to \mathbb{R}$ be defined by

$$F(x, y) = r(x, f(x) - y),$$

which is concave and continuously differentiable. With F, the Bellman equation is written as

$$v^*(x) = \sup_{y \in \Gamma(x)} F(x, y) + \beta v^*(y) \qquad (x \in X). \tag{3}$$

Now assume that for any $x_0 > 0$, the supremum in (2) is attained by $c^* = c^*(x_0)$ with $0 < c^* < f(x_0)$. Then the supremum in (3) is attained by $y^* = f(x_0) - c^*$, where $0 < y^* < f(x_0)$. Clearly, $(x_0, y^*) \in \operatorname{int} A$ (by the continuity of f), so that condition (b) in Corollary 3 holds, and hence, by Corollary 3, v^* is differentiable at any $x_0 > 0$, and we have

$$v^{*\prime}(x_0) = F_1(x_0, y^*) = r_1(x_0, f(x_0) - y^*) + r_2(x_0, f(x_0) - y^*) f'(x_0). \tag{4}$$

On the other hand, once we have the differentiability of v^*, by the standard envelope theorem argument, such as in Hotelling's or Shephard's lemma or whatever, applied to the Bellman equation (2), we have

$$v^{*\prime}(x_0) = r_1(x_0, c^*) + \beta v^{*\prime}(f(x_0) - c^*) f'(x_0). \tag{5}$$

Of course, the envelope theorem formulas (4) and (5) are equivalent, through the first-order condition for the maximization in the right hand side of (2), that c^* satisfy

$$r_2(x_0, c^*) - \beta v^{*\prime}(f(x_0) - c^*) = 0. \tag{6}$$

By this condition, (5) leads to (4).

References

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