



# Mean-field approximation of forward-looking population dynamics <sup>☆</sup>

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## ABSTRACT

We study how the equilibrium dynamics of a continuum-population game approximate those of large finite-population games. New agents stochastically arrive to replace exiting ones and make irreversible action choices to maximize the expected discounted lifetime payoffs. The key assumption is that they only observe imperfect signals about the action distribution in the population. We first show that the stochastic process of the action distribution in the finite-population game is approximated by its mean-field dynamics as the population size becomes large, where the approximation precision is uniform across all equilibria. Based on this result, we then establish continuity properties of the equilibria at the large population limit. In particular, each agent becomes almost negligible, in the sense that in equilibrium, each agent's action is almost optimal against the (incorrect) belief that it has no impact on others' actions as presumed in the continuum-population case. Finally, for binary-action supermodular games, we show that when agents are patient, there is a unique equilibrium as observation noise becomes small while the population size becomes large. In this equilibrium, every agent chooses a risk-dominant action, and the population globally converges to the corresponding steady state.

## 1. Introduction

### 1.1. Motivation and overview

Many economic environments involve interactions among a large number of agents. A typical modeling approach to study the dynamics of their behavior is by a continuum-population model, which presumes each individual agent to be negligible. There it is also often presumed that agents' idiosyncratic randomness is canceled out and the aggregate behavior evolves continuously and deterministically following mean-field dynamics. These features are useful for tractability in various economic applications, such as search markets (e.g., Diamond and Fudenberg, 1989), industrialization (e.g., Matsuyama, 1991), spatial agglomeration (e.g., Krugman, 1991a), and traffic congestion (e.g., Smith, 1984), among others.

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Continuum population is viewed as an idealization of a large but finite population in the real world. We want to have a formal foundation that guarantees that the predictions of the former are derived as the limit of finite models as the number of agents becomes large. As a stylized model, we consider a class of overlapping-population dynamics with  $N$  symmetric agents. New agents stochastically arrive according to independent Poisson processes to replace exiting ones and make irreversible action choices upon arrival, to maximize the expected discounted lifetime payoffs which depend on the action distribution in the population, or the population state. Our goal is to establish that, under certain conditions, the dynamics of a finite-population game is approximated by the mean-field dynamics of the continuum-population game as the population size  $N$  becomes large.

Such approximation results have been obtained in the evolutionary game theory literature, where the dynamics is induced by an exogenously fixed rule of *myopic* action adaptation (e.g., Boylan, 1995; Binmore et al., 1995; Corradi and Sarin, 2000; Benaïm and Weibull, 2003, 2009; Sandholm, 2003). They, however, do not directly apply to settings where agents' *forward-looking* expectations are of significant importance and their behavior is endogenously determined in equilibrium, as in many applications (e.g., Krugman, 1991b; Matsuyama, 1991). First, in a finite population, an individual action change, if perfectly observed, may lead to a large change in the actions of other agents. In a public goods situation, for example, an observed deviation from cooperation may cause a chain of punishment by others, which is assumed away in a continuum-population case. Therefore, with forward-looking agents, the equilibria may exhibit a discontinuity at the large population limit. Second, in the case of forward-looking agents, there can be multiple dynamic equilibria for a given initial population state, for example in coordination games, and in principle, how large the population size  $N$  must be in order to obtain mean-field approximation may depend on the equilibrium in consideration. Thus we want to obtain approximation whose precision is uniform across all equilibria.

Our key assumption in this paper is that each agent only observes an imperfect signal of the population state, and the distribution of the signal changes Lipschitz continuously with respect to the population state in the total variation norm. This assumption is satisfied, for example, when the signal is of additive form (state plus noise) where the noise term admits a Lipschitz continuous density on its compact support.

Under this assumption, we first show that the stochastic process of the population state induced by agents' strategy profile in a large finite game is approximated by the associated mean-field dynamics. The mean-field dynamics is described by an ordinary differential equation, and importantly, the precision of our approximation is uniform across all strategies, hence across all equilibria. Our imperfect observation assumption implies that agents' choices are smoothed out in a way that the mean-field dynamics are uniformly Lipschitz continuous across all strategy profiles. This allows us to obtain mean-field approximation, using the technique of Benaïm and Weibull (2003), with a uniform precision.

Building on this uniform approximation result, we then derive several continuity properties of the set of equilibria of large finite-population games at the large population limit. In particular, we obtain an "agent smallness" result, that as the population size becomes large, each agent becomes almost negligible; each equilibrium action is nearly optimal under the (incorrect) belief that her action will have no impact on the population state. For example, this implies that agents never choose dominated actions when the population size  $N$  is sufficiently large, as in the continuum-population game. This is in contrast with the case of perfect observation, where dominated actions (e.g., cooperation in a public goods game) may be played in equilibrium, even with a large number of agents. Under an additional regularity condition on the signal structure, we also show that the set of equilibria of large finite-population games is approximated by that of the mean-field game and vice versa (modulo  $\epsilon$ -optimality).

Finally, we study the double limit as the population size becomes large first and then the observation noise becomes small. We specialize to binary-action supermodular games with an additive form of observation noise. We show that, for sufficiently patient agents, there is a unique equilibrium as the noise level becomes small while the population size becomes large. In this equilibrium, all incoming agents always choose a generalization of risk-dominant action (the "Laplacian" action), so that the population state globally converges to the corresponding unanimous steady state. This strategy is also an equilibrium in a continuum-population model with perfect observation, but such a model also admits many other equilibria (Matsuyama, 1991; Matsui and Matsuyama, 1995). Our result demonstrates that equilibrium multiplicity disappears under an appropriate approximation with small noise. We illustrate the result by means of simple economic examples, such as market thickness and industrialization.

## 1.2. Related literature

The idea of approximating a finite-population model with a continuum-population model has long been of major interest in economics, tracing back to the literature on competitive equilibrium (e.g., Debreu and Scarf, 1963; Aumann, 1964). This paper formalizes such an idea in a setting in which forward-looking agents make irreversible action choices, and in the continuum-population limit, the population action distribution evolves continuously over time. This setting is also employed in the class of dynamic population models sometimes called perfect foresight dynamics (Matsuyama, 1991; Matsui and Matsuyama, 1995) and studied extensively in the context of equilibrium selection in games<sup>1</sup> and applied to various economic applications.<sup>2</sup> The current paper can be seen as offering a finite-population foundation for this class of models.<sup>3</sup> Moreover, we show that introducing a small amount of observational noise into these models can substantially refine the prediction.

<sup>1</sup> See also, e.g., Hofbauer and Sorger (1999); Kojima (2006); Oyama et al. (2008); Takahashi (2008); Iijima (2015).

<sup>2</sup> See also, e.g., Matsuyama (1992a,b); Kaneda (1995); Matsuyama and Takahashi (1998); Oyama (2009).

<sup>3</sup> Weintraub et al. (2011) study a model of industry dynamics (with reversible choices) and show that under certain conditions, stationary equilibria of the continuum-population version, where the aggregate state is constant over time, approximate Markov perfect equilibria of the finite-population version as the number of firms becomes large.

As mentioned earlier, the evolutionary game literature has studied approximation of finite-population dynamics by their mean field; see Sandholm (2010) for a textbook treatment. In this literature, the presence of payoff shocks is often assumed to guarantee the continuity of the mean-field dynamics.<sup>4</sup> Assuming payoff shocks, instead of observation noise, would not work in our framework with forward-looking agents. In that case, the continuation payoffs would in general remain discontinuous in the population state.

The literature on repeated games has shown that under certain conditions, individual actions become negligible with a large number of agents. It has been shown that in repeated games with *simultaneous* moves, only repeated static Nash equilibria are supported in equilibrium as the number of agents becomes large when the monitoring technology is imperfect (e.g., Green, 1980; Sabourian, 1990; Levine and Pesendorfer, 1995; Fudenberg et al., 1998; Kalai and Shmaya, 2018). The observation noise assumption in our paper plays a similar role to imperfect monitoring in these models. Our aim, however, is to establish mean-field approximation of dynamics, which has no analogue in this literature. In particular, we consider a model with *asynchronous* moves—to capture irreversibility in individual decisions, a prominent feature in applications such as choices of technologies or industrial sectors—where the corresponding mean-field dynamics generates a continuous path of action distributions.

Our equilibrium uniqueness result for binary-action supermodular games is related to the findings in the global game literature (e.g., Carlsson and van Damme, 1993; Frankel et al., 2003; Morris and Shin, 2003). Papers in this literature establish uniqueness results in static incomplete information games in which each agent observes a noisy signal about the payoff relevant state before choosing actions. In contrast, our model is dynamic and features no payoff uncertainty. Yet, the unique action obtained in our model coincides with that selected in global games, the Laplacian action as in Morris and Shin (2003)—the best response action against the uniform belief over the actions in the population. Burdzy et al. (2001) and Frankel and Pauzner (2000) also obtain similar equilibrium uniqueness results based on arguments analogous to those of global games, using a dynamic model with asynchronous moves and a stochastic payoff state. Agents in these models perfectly observe the payoff state as well as the action distribution. In contrast, agents' action choice in our model is uniquely determined without stochastic payoff fluctuations. While the logic is different behind these results, there is some type of “friction” in each model that makes it harder for the agents to coordinate behavior, enough to eliminate all equilibria but one—the Laplacian equilibrium: combined with action asynchronicity, it is stochasticity of payoffs in the dynamic global games of Burdzy et al. (2001) and Frankel and Pauzner (2000) and imperfect observation about the action distribution in our model.

Blume (2005) considers a binary-action random-matching coordination game played by a finite population of infinitely lived agents in a setting similar to ours but without observation noise. He shows that if agents are sufficiently patient, then there is a unique Markov equilibrium, in which players always choose the risk-dominant action.<sup>5</sup> This result relies on the feature of his model that each individual action is *not* negligible, and in fact, there would be a discontinuity at the continuum-population limit of his model, where equilibrium multiplicity would arise in contrast to our setting.

## 2. Finite-population model

### 2.1. Environment

We consider an overlapping generation game model parameterized by the population size  $N \in \mathbb{N}$ .<sup>6</sup> Time runs over an infinite horizon  $\mathbb{T} = [0, \infty)$ . Each agent in the population is endowed with an independent and identical Poisson clock, with its arrival rate normalized to be 1. Whenever a shock arrives at an agent, that agent exits the population and is replaced by a newborn agent. A newborn agent makes an irreversible action choice upon entry to maximize her expected discounted lifetime payoffs as formulated below.<sup>7</sup>

The finite set of actions, which is common to all the agents, is denoted by  $A$ . We let  $\Delta \subset \mathbb{R}^{|A|}$  denote the probability simplex over  $A$ .<sup>8</sup> For  $x \in \Delta$ ,  $x_a$  denotes the fraction of action  $a \in A$  under  $x$ , and  $e_a \in \Delta$  denotes the unit vector that assigns 1 to the  $a$ -coordinate. Let  $x_N(t) \in \Delta_N$  denote the *state* (the action distribution in the population) at time  $t$ , where

$$\Delta_N = \left\{ x \in \Delta \mid x_a = 0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1 \text{ for all } a \in A \right\}.$$

At each time  $t \in \mathbb{T}$ , each agent receives a flow payoff depending on the aggregate state at  $t$  as well as her own (committed) action, where she discounts the future payoffs with the common discount rate  $r > 0$ . The flow payoffs are given by the common function  $u : A \times \Delta \rightarrow \mathbb{R} : u(a, x)$  is the payoff to action  $a \in A$ , which is assumed to be a Lipschitz continuous function of the current state  $x \in \Delta$ .<sup>9</sup> Thus, an agent, if entering the society at time  $t \in \mathbb{T}$  with action  $a$ , receives a (normalized) discounted total payoff

<sup>4</sup> Gorodeisky (2009) and Roth and Sandholm (2013) take an alternative approach without continuity of the mean-field, using differential inclusions instead of differential equations.

<sup>5</sup> See also Ambrus and Ishii (2015) for related results.

<sup>6</sup> While we employ a single-population model for simplicity, it would be straightforward to consider a multi-population version.

<sup>7</sup> An alternative setting would be to assume that each agent exits the population without replacement and a newborn agent enters with an independent Poisson clock with arrival rate  $N$ . In this case, the population size fluctuates over time around  $N$ . We conjecture that our main results hold under this setting as well.

<sup>8</sup> We let  $\|\cdot\|$  denote the sup norm in a finite-dimensional space.

<sup>9</sup> Our results hold even when the payoff function depends on  $N$ , as written as  $u_N$ , assuming that there exists a uniformly convergent limit  $\lim_{N \rightarrow \infty} u_N(a, x)$  that is Lipschitz continuous in  $x \in \Delta$ .

$$(1+r) \int_t^\tau e^{-r(t'-t)} u(a, x_N(t')) dt'$$

if she exits at time  $\tau$ , where the value is normalized by the effective discount rate  $1+r$  (rate of exit plus rate of time preference).

We assume that, upon entry, agents cannot observe the calendar time  $t$  and only imperfectly observe the state  $x$  through a noisy signal. The latter—imperfectness of state observability—is our substantial assumption and will be formalized in Section 2.3. Previously, the next Section 2.2 examines the perfect observation benchmark case.

### 2.2. Benchmark examples

We first consider the benchmark case in which agents perfectly observe the population state. We provide two examples to illustrate issues in establishing large population approximation in our model with forward-looking agents.

Both examples have binary actions  $A = \{0, 1\}$  (and thus the state is denoted  $x = (x_0, x_1)$ ). The first example demonstrates the possibility that an individual action can have a non-negligible impact even with large  $N$  in the perfect observation version of our game.

**Example 1 (Public goods).** Consider a public good game in which action 1 corresponds to providing public goods and action 0 corresponds to free riding. Suppose that the payoffs take the form  $u(0, x) = x_1$  and  $u(1, x) = x_1 - c$ , where  $c > 0$  denotes the cost for public good provision. The dominated action 1 would not be played if agents were negligible in the population (so that each agent’s action choice would not influence  $x$  and others’ behavior). When the state is perfectly observed, this is not the case. In particular, if  $c < \frac{1}{2+r}$ , then there exists  $N_0$  such that for all  $N \geq N_0$ , there is an equilibrium in which action 1 is chosen if (and only if) all the agents in the population are observed to use action 1; see Appendix B.1.1 for the details. That is, in this equilibrium a free-riding agent is “punished” immediately, which provides an incentive to provide public goods.  $\diamond$

The next example highlights equilibrium multiplicity.

**Example 2 (Coordination game).** Consider a coordination game with  $u(0, x) = c$  and  $u(1, x) = x_1$ , where  $c \in (0, 1)$ . Note that the myopic best response action is 1 (resp. 0) if  $x_1 > c$  (resp.  $x_1 < c$ ). As we verify in Appendix B.1.2, there can be many equilibria in this game when agents perfectly observe the state. In particular, for each  $\kappa \in [0, 1]$ , consider the  $\kappa$ -strategy, where action 1 is chosen if and only if at least fraction  $\kappa$  of the agents in the population play action 1. If  $c > \frac{1}{2+r}$ , then there exist a non-trivial interval  $\mathcal{K} \subset [0, 1]$  and  $N_0$  such that if  $N \geq N_0$ , then the  $\kappa$ -strategy is an equilibrium for all  $\kappa \in \mathcal{K}$  (see Proposition B.3 for an exhaustive characterization of equilibria under  $\kappa$ -strategies with large  $N$ ). Thus, in this case, there are unboundedly many equilibria as  $N$  becomes large.  $\diamond$

Existing studies in the evolutionary game literature have established that the population dynamics under a fixed strategy can be approximated by its mean-field dynamics as  $N \rightarrow \infty$ : the strategy is uniquely determined (except at indifferences) once one focuses on agents who play myopic best responses (or follow some other fixed behavior rule such as imitation). In the case of forward-looking agents, there can be multiple equilibria, and in principle, how large  $N$  must be in order to obtain mean-field approximation may depend on the equilibrium strategy in consideration. Moreover, the number of equilibrium strategies can grow unboundedly as  $N$  becomes large (as in Example 2). Thus we want to obtain approximation whose precision is uniform across all equilibrium strategies.

As we will show in the next section, the presence of observation noise (to be formalized as Assumption L below) allows us to deal with the above issues. In addition, we will show in Section 4 that an equilibrium is unique for binary-action supermodular games for a small amount of observation noise, in contrast with the equilibrium multiplicity in Example 2.

### 2.3. Imperfect observation

We now introduce our main assumption: upon entry, the newborn agent cannot directly observe the state  $x$ , but only receives a noisy signal about  $x$ , conditional on which she chooses an action from  $A$ .<sup>10</sup> The signal is drawn from a probability distribution  $\mu_x$  on a (Borel) subset  $Z$  of  $\mathbb{R}^m$  for some  $m$ , where  $\mu_x$  depends only on the current state  $x \in \Delta$ .<sup>11</sup> We assume that either (i)  $Z$  is finite, or (ii)  $Z$  has positive Lebesgue measure, and for each  $x \in \Delta$ ,  $\mu_x$  admits a bounded density  $f_x$ . We also assume that  $\bigcup_{x \in \Delta_N} \text{supp}(\mu_x) = Z$  for all large enough  $N$ . Hereafter we restrict attention to such large  $N$ ’s. The agents share the same (exogenous) prior  $\nu_N$  over  $\Delta_N$ .<sup>12</sup> A strategy is a measurable function  $\sigma : Z \rightarrow \Delta$  that associates to each signal  $z \in Z$  a probability distribution  $\sigma(z) \in \Delta$  over actions to be committed to. Let  $\Sigma$  denote the set of all strategies. We will only consider symmetric strategy profiles and refer to the symmetric profile where all agents adopt a common strategy  $\sigma$  simply as  $\sigma$ . Given a strategy  $\sigma \in \Sigma$ , let  $E\sigma : \Delta \rightarrow \Delta$  denote its aggregate strategy, which is defined by

<sup>10</sup> Here the new agent observes a signal about the action distribution in  $\Delta_N$  that includes the action of the agent who is to be replaced. An alternative formulation is to observe a signal about the action distribution in  $\Delta_{N-1}$  that excludes that agent’s action. Our main results hold under this formulation as well.

<sup>11</sup> Whether signals can be correlated across agents is irrelevant for the analysis, since almost surely only at most one agent receives a signal at each time.

<sup>12</sup> In Section 5, we discuss how results generalize under other specifications of beliefs.

$$(E\sigma)(x) = \int \sigma(z) d\mu_x(z)$$

for each  $x \in \Delta$ , where  $(E\sigma)(x) \in \Delta$  is the distribution of actions taken by a new agent given the current state  $x$ .

The initial state  $x_N(0) = x_0 \in \Delta_N$  is randomly drawn according to an exogenous distribution on  $\Delta_N$ . A strategy  $\sigma$  when commonly employed by the agents induces a stochastic process of the state variable  $\{x_N(t, x_0, \sigma)\}_{t \in \mathbb{T}}$ . We will sometimes simplify the notation by suppressing the dependence on  $x_0$ .

For an agent who commits to action  $a \in A$ , her (normalized) continuation payoff  $V_N(a, x, \sigma)$  conditional on the state at time  $t$  being  $x_N(t, \sigma) = x$  with  $x_a \geq \frac{1}{N}$  when the other agents adopt strategy  $\sigma$  is determined by the following recursive equation:

$$(1+r)V_N(a, x, \sigma) = (1+r)u(a, x) + (N-1) \left( \sum_{a', a'' \in A} (E\sigma)(a' | x) x_{a'}^{-a} V_N \left( a, x + \frac{e_{a'} - e_{a''}}{N}, \sigma \right) - V_N(a, x, \sigma) \right), \quad (2.1)$$

where  $x_{a''}^{-a}$  is the  $a''$ -th component of  $x^{-a} := \frac{Nx - e_a}{N-1} \in \Delta_{N-1}$ , and the second term represents the expected instantaneous change in  $V_N$  through an adjustment of other agents' actions, which occurs with arrival rate  $N-1$ .

Let  $v_{N,z}$  be the posterior beliefs about the state conditional on the observation of a signal  $z$  by a new agent. For every  $z$ , we assume that  $v_{N,z}$  is well defined and that its weak-convergent limit  $v_z := \lim_{N \rightarrow \infty} v_{N,z}$  exists. For each observed signal  $z \in Z$ , the agent chooses an action  $a \in A$  that maximizes the normalized expected discounted payoff

$$W_N(a, z, \sigma) = \int_{\Delta_N} \sum_{a' \in A} x_{a'} V_N \left( a, x + \frac{e_a - e_{a'}}{N}, \sigma \right) d v_{N,z}(x). \quad (2.2)$$

The agent's choice of action  $a$  changes the current state  $x$  by  $\frac{e_a - e_{a'}}{N}$ , where  $a'$  is the action of her predecessor, which is distributed according to  $x$ . Given signal  $z$ , the agent uses posterior belief  $v_{N,z}$  about  $x$  to compute the expected payoff. We write  $W_N(z, \sigma) = (W_N(a, z, \sigma))_{a \in A}$  and  $V_N(x, \sigma) = (V_N(a, x, \sigma))_{a \in A}$ .

We denote, with an emphasis on the dependence on the population size  $N$ , this finite-population game as described above by  $\Gamma_N$ . We are interested in symmetric equilibria of  $\Gamma_N$  defined by the following:

**Definition 1.** A strategy  $\sigma \in \Sigma$  is an *equilibrium* of  $\Gamma_N$  if for all  $a \in A$ ,

$$\sigma(a|z) > 0 \Rightarrow a \in \arg \max_{a' \in A} W_N(a', z, \sigma)$$

for almost all  $z \in Z$ .<sup>13</sup>

In this definition, we assume equilibrium expectations about the play of the future generations, but exogenous inferences about the past play. That is, in an equilibrium  $\sigma$ , each incoming agent optimizes against the correct belief about  $\sigma$  itself, while the belief  $v_{N,z}$  about the current population state  $x$  given signal  $z$  is updated from an exogenously fixed prior. Our motivation behind this formulation is to incorporate forward-looking incentives as a minimal modification to evolutionary games. That literature often studies myopic agents who have limited knowledge about the current state and adopt certain exogenous inference rules.<sup>14</sup> Here we only replace myopia by rational expectations. In Section 5.2, we will discuss how our results would be extended under alternative formulations.

## 2.4. Uniform noise structure

From now on, we impose the following restriction on signal distributions  $(\mu_x)_{x \in \Delta}$ , which will play a key role in our analysis. Let  $\|\cdot\|_{TV}$  denote the total variation norm.<sup>15</sup>

**Assumption L.** There exists  $L > 0$  such that  $\|\mu_x - \mu_{x'}\|_{TV} \leq L\|x - x'\|$  for all  $x, x' \in \Delta$ .

That is, a small change in the state leads to a small change in the noise distribution, where the rate of change is bounded globally by a constant  $L$ . An immediate consequence of this assumption is that individual choices are smoothed out so that the aggregate strategy  $E\sigma$  is Lipschitz continuous in the population state  $x$  uniformly over  $\sigma$ .

**Lemma 1.**  $\|(E\sigma)(x) - (E\sigma)(x')\| \leq L\|x - x'\|$  for all  $\sigma \in \Sigma$  and  $x, x' \in \Delta$ .

**Proof.** Note that  $|\int f(z) d\mu_x(z) - \int f(z) d\mu_{x'}(z)| \leq \|\mu_x - \mu_{x'}\|_{TV}$  for any measurable function  $f$  with  $\sup_z |f(z)| \leq 1$ . Thus the claim follows by applying this to  $\sigma(a|z)$  for each  $a$ .  $\square$

<sup>13</sup> By "almost all  $z \in Z$ ", we mean (i) "all  $z \in Z$ " in the case of finite  $Z$  and (ii) "all  $z \in Z$  except for a set with Lebesgue measure 0" in the case of infinite  $Z$ .

<sup>14</sup> See, e.g., Hurkens (1995); Sandholm (2001, 2003), Oyama et al. (2015), and Salant and Cherry (2020).

<sup>15</sup> That is,  $\|\mu - \mu'\|_{TV} = 2 \sup_B |\mu(B) - \mu'(B)|$  where the supremum is taken over all the (Borel) measurable subsets  $B$  of  $Z$ .

The following are simple examples of signal structures that satisfy Assumption L.

**Example 3 (Sampling).** Each agent draws a size- $k$  sample (with replacement) of agents in the population and observes the distribution of actions of those agents. Thus we have  $Z = \Delta_k$ , which is a finite set, and

$$\mu_x(\{z\}) = \left( \binom{k}{(kz_a)_{a \in A}} \prod_{a \in A} x_a^{kz_a} \right)$$

for each  $z \in \Delta_k$ . For each  $z \in \Delta_k$ ,  $\mu_x(\{z\})$  is Lipschitz continuous in  $x$  on  $\Delta$  with a coefficient, say,  $L_z$ . Therefore,  $\|\mu_x - \mu_{x'}\|_{TV} = \sum_{z \in \Delta_k} |\mu_x(\{z\}) - \mu_{x'}(\{z\})|$  is bounded above by  $L\|x - x'\|$  with  $L = \sum_{z \in \Delta_k} L_z$ .  $\diamond$

**Example 4 (Additive noise).** Write  $A = \{0, \dots, |A| - 1\}$ . Each agent observes a signal of the form  $z = x_{-0} + \eta\gamma$  at each state  $x$ , where  $x_{-0} = (x_1, \dots, x_{|A|-1})$ ,  $\eta > 0$ , and  $\gamma$  follows a distribution that admits a density  $g : \mathbb{R}^{|A|-1} \rightarrow \mathbb{R}$  that is supported on a convex and compact set and is Lipschitz continuous on the support. Then Assumption L is satisfied in this setting; see Appendix B.2.1 for the details.  $\diamond$

We confirm that our game indeed has an equilibrium:

**Proposition 1.** For any  $N$ , the game  $\Gamma_N$  has an equilibrium.

The proof of this proposition, as well as those of subsequent results, can be found in the Appendix. Throughout the rest of this paper, let  $\Sigma_N^*$  denote the set of equilibria in the  $N$ -agent game  $\Gamma_N$ .

### 3. Large population limit

#### 3.1. Mean-field model

We introduce the associated mean-field game, which is denoted by  $\Gamma$ . The setting is the same as the  $N$ -agent game except that there are a continuum of agents in the population, as in the model of perfect foresight dynamics (Matsuyama, 1991; Matsui and Matsuyama, 1995). As in the  $N$ -agent game, a strategy is a measurable function  $\sigma : Z \rightarrow \Delta$ , which is identified with the symmetric strategy profile where all the agents adopt this strategy  $\sigma$ .

For any  $\sigma \in \Sigma$  and  $x \in \Delta$ , define the vector  $F^\sigma(x) = (F_a^\sigma(x))_{a \in A} \in \mathbb{R}^{|A|}$  whose component is given by

$$F_a^\sigma(x) = (E\sigma)(a|x) - x_a,$$

where the inflow term  $(E\sigma)(a|x)$  represents the fraction of new agents who choose action  $a$ , and the outflow term  $x_a$  represents the fraction of agents playing action  $a$  who exit. If all the agents adopt strategy  $\sigma$ , the state evolves according to the *mean-field dynamics*,

$$\dot{x}(t) = F^\sigma(x(t)), \quad x(0) = x_0, \tag{3.1}$$

where  $x_0 \in \Delta$  is the exogenously given initial state. Due to the Lipschitz continuity of  $E\sigma$  (Lemma 1), the dynamics defined by (3.1) admits a unique Lipschitz continuous path  $\phi(\cdot, x_0, \sigma) : \mathbb{T} \rightarrow \Delta$  for each initial condition  $x_0$ , which we call the *mean-field path* from  $x_0$ . Moreover,  $\phi(t, x_0, \sigma)$  is continuous in the initial state  $x_0$ .

In this game, agents optimize against the mean-field path under the premise that their own actions have no impact. Thus, each entering agent, observing a signal  $z$ , chooses an action that maximizes

$$W(a, z, \sigma) = \int_{\Delta} V(a, x, \sigma) d\nu_z(x)$$

with

$$V(a, x, \sigma) = (1+r) \int_0^\infty e^{-(1+r)s} u(a, \phi(s, x, \sigma)) ds$$

expressing the (normalized) continuation payoff given strategy  $\sigma$ , where  $\phi(\cdot, x, \sigma)$  is the mean-field path from  $x$ .

**Definition 2.** A strategy  $\sigma \in \Sigma$  is an *equilibrium* of  $\Gamma$  if for all  $a \in A$ ,

$$\sigma(a|z) > 0 \Rightarrow a \in \arg \max_{a' \in A} W(a', z, \sigma)$$

for almost all  $z \in Z$ .

Equilibrium existence is guaranteed as in the  $N$ -agent game:

**Proposition 2.** *There exists an equilibrium of  $\Gamma$ .*

Let  $\Sigma^*$  denote the set of equilibria of  $\Gamma$ .

The main task of this paper is to formulate the sense in which the equilibrium dynamics of the mean-field model is a good approximation of the equilibrium stochastic process of the state of the finite-agent model.

### 3.2. Mean-field approximation

The following lemma shows that the mean-field dynamics approximates the stochastic process  $x_N$  induced by the  $N$ -agent game  $\Gamma_N$ .

**Lemma 2.** *For any  $\epsilon > 0$  and  $T > 0$ , there exist  $C > 0$  and  $\bar{N}$  such that*

$$\Pr \left[ \sup_{t \in [0, T]} \|x_N(t, x_0, \sigma) - \phi(t, x_0, \sigma)\| \geq \epsilon \mid x_N(0) = x_0 \right] \leq e^{-CN}$$

*holds for any  $\sigma \in \Sigma$ ,  $N \geq \bar{N}$ , and  $x_0 \in \Delta_N$ .*

For given deviation bound  $\epsilon$  and horizon length  $T$ , the lemma shows that the process  $x_N(\cdot)$  under the finite-population model with large  $N$  is approximated by a mean-field path. Importantly, approximation holds uniformly across all strategies  $\sigma \in \Sigma$ ; this is essential given the possibility of equilibrium multiplicity with forward-looking agents. The proof makes use of the stochastic approximation technique in Benaïm and Weibull (2003), who consider myopic evolutionary dynamics. Their result implies that the above approximation holds under *fixed* strategy  $\sigma$ , so that  $\bar{N}$  and  $c$  may depend on  $\sigma$ . A key part of our proof uses the uniform Lipschitz continuity of  $E\sigma$ , and hence of  $F^\sigma$ , due to Assumption L (Lemma 1) to establish approximation uniformly across all strategies  $\sigma$ .<sup>16</sup>

Now the following holds as a corollary of this result, which shows that agents' value functions can be approximated by the mean-field model as  $N \rightarrow \infty$ , uniformly across all strategies. This result is convenient for analyzing agents' incentives under large  $N$  with the more tractable mean-field model.

**Lemma 3.** *For any  $\epsilon > 0$ , there exists  $\bar{N}$  such that*

$$|V_N(a, x, \sigma) - V(a, x, \sigma)| \leq \epsilon$$

*holds for any  $\sigma \in \Sigma$ ,  $N \geq \bar{N}$ ,  $x \in \Delta_N$ , and  $a \in A$ .*

### 3.3. Smallness of agents

Recall that each agent's action is assumed to be negligible in the mean-field model. Below we show that this property holds approximately in our  $N$ -agent model for sufficiently large  $N$ .

**Definition 3.** For  $\epsilon > 0$ , an equilibrium  $\sigma \in \Sigma_N^*$  is said to satisfy the  $\epsilon$ -small agent property if for almost all  $z \in Z$  and all  $a^* \in A$  such that  $\sigma(a^*|z) > 0$ ,

$$\int_{\Delta_N} V_N(a^*, x, \sigma) d\nu_{N,z}(x) \geq \int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x) - \epsilon$$

holds for all  $a \in A$ .

This property requires that taking  $a^*$  be  $\epsilon$ -optimal under the hypothesis that the agent's action does not change the current state  $x$ , i.e., she is negligible in the population. Recall the definition of (exact) equilibrium in Section 2, where the objective  $W_N$  defined as (2.2) takes into account that choosing  $a^*$  changes the state by  $\frac{e_{a^*} - e_{a'}}{N}$ .

**Theorem 1.** *For any  $\epsilon > 0$ , there exists  $\bar{N}$  such that for all  $N \geq \bar{N}$ , all equilibria of  $\Gamma_N$  satisfy the  $\epsilon$ -small agent property.*

That is, with sufficiently large  $N$ ,  $W_N(a, z, \sigma)$  is approximated by  $\int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x)$ , uniformly over  $\sigma$ . This result builds on Lemma 3, which allows us to approximate the continuation value of the finite-population model by that of the mean-field model. The latter,  $V(a, x, \sigma)$ , is easily verified to be uniformly Lipschitz continuous in the current state  $x$ , and hence the individual influence on  $x$ , which is of order  $\frac{1}{N}$ , is negligible for large  $N$ .

<sup>16</sup> Indeed, Lemma 2 fails without Assumption L. For instance, consider Example 2, where the state is perfectly observable so that Assumption L does not hold, and assume  $c > \frac{1}{2+r}$  and  $N \geq N_0$ . For the  $\kappa$ -strategy equilibria as described (assume that  $\kappa$  is a multiple of  $\frac{1}{N}$ ), the process  $x_N(\cdot)$  if starting with  $x_{N,1}(0) = \kappa$  is inherently random, branching with non-trivial probabilities into different basins of attraction and hence cannot be approximated by a deterministic path, even when  $N$  is large.

We say that action  $a$  is *dominated* if there is an action  $\bar{a}$  such that  $u(\bar{a}, x) > u(a, x)$  for all  $x$ . A corollary of Theorem 1 is that, when the population size is large, agents will not play dominated actions in any equilibrium.

**Corollary 1.** *There exists  $\bar{N}$  such that if  $N \geq \bar{N}$ , then for any equilibrium  $\sigma \in \Sigma_N^*$  and any dominated action  $a$ ,  $\sigma(a|z) = 0$  holds for almost all  $z$ .*

Compare Example 1, where a dominated action is played in an equilibrium under perfect observation.

### 3.4. Continuity properties of equilibria

The results below allow us to approximate the equilibria of the  $N$ -agent game by the equilibria of the mean-field game.

**Proposition 3.** *For any sequence  $(\sigma_N)_{N \in \mathbb{N}}$  where  $\sigma_N \in \Sigma_N^*$  for each  $N$ , there exists  $\sigma \in \Sigma^*$  and a subsequence  $(\sigma_{N_k})_{k \in \mathbb{N}}$  such that for all  $x \in \Delta$ ,*

$$\lim_{k \rightarrow \infty} \phi(t, x, \sigma_{N_k}) = \phi(t, x, \sigma)$$

*uniformly on compact time intervals.*

The above result establishes upper semi-continuity of equilibrium at the large population limit. Under an additional assumption on  $v_{N,z}$  (Assumption U below), we also obtain a continuity result by relaxing optimality to  $\epsilon$ -optimality.

**Definition 4.**  $\sigma \in \Sigma$  is an  $\epsilon$ -equilibrium of  $\Gamma_N$  if for almost all  $z \in Z$ ,

$$\sigma(a|z) > 0 \Rightarrow W_N(a, z, \sigma) \geq W_N(a', z, \sigma) - \epsilon \text{ for all } a' \in A;$$

likewise,  $\sigma \in \Sigma$  is an  $\epsilon$ -equilibrium of  $\Gamma$  if for almost all  $z \in Z$ ,

$$\sigma(a|z) > 0 \Rightarrow W(a, z, \sigma) \geq W(a', z, \sigma) - \epsilon \text{ for all } a' \in A.$$

Let  $\Sigma_N^{*,\epsilon}$  (resp.  $\Sigma^{*,\epsilon}$ ) denote the set of  $\epsilon$ -equilibria of  $\Gamma_N$  (resp.  $\Gamma$ ). Now we impose the following assumption, which requires weak convergence  $v_{N,z} \rightarrow v_z$  to be uniform in  $z \in Z$ :

**Assumption U.** *For any continuous function  $v : \Delta \rightarrow \mathbb{R}$ ,  $\int_{\Delta_N} v(x) d v_{N,z}(x) \rightarrow \int_{\Delta} v(x) d v_z(x)$  as  $N \rightarrow \infty$  uniformly on  $Z$ .*

This assumption obviously holds when  $Z$  is a finite set, in particular in Example 3. It also holds in the special case of Example 4 as discussed in Example 5 below.

The next theorem shows that, under this assumption, the equilibria of  $\Gamma_N$  with large  $N$  can be approximated by those of  $\Gamma$  (and vice versa), as long as we allow for  $\epsilon$ -optimality.

**Theorem 2.** *Suppose that Assumption U holds. Then for any  $\epsilon > 0$ , there exists  $\bar{N}$  such that for any discount rate  $r > 0$ ,  $\Sigma_N^* \subset \Sigma^{*,\epsilon}$  and  $\Sigma^* \subset \Sigma_N^{*,\epsilon}$  for all  $N \geq \bar{N}$ .*

In fact, in the binary-action supermodular case considered in the next section, these sets coincide for some  $\epsilon > 0$  (in particular,  $\Sigma_N^* = \Sigma^*$ ) for all sufficiently large  $N$ , if the observation noise is as given in the following.

**Example 5 (Additive noise).** Assume additive noise in the signals as in Example 4. Further assume that the priors  $v_N$  in the  $N$ -agent models converge weakly as  $N \rightarrow \infty$  to the prior  $v$  in the mean-field model and that  $v$  admits a density strictly positive and continuous on  $\Delta$ . Then Assumption U is satisfied in this setting; see Appendix B.2.2 for the details.  $\diamond$

## 4. Uniqueness in binary-action supermodular games with small noise

Our analysis so far has focused on taking the population size to be large given a fixed observation noise structure. In this section, we study the double limit as the population becomes large and then the noise becomes small. Specifically, we focus on binary-action supermodular games with the form of additive noise in Example 5. We show that, for sufficiently patient agents, there is a unique equilibrium as the noise level becomes small while the population size becomes large.

Let  $A = \{0, 1\}$ . For notational simplicity we identify the state space  $\Delta$  with the interval  $[0, 1]$ , so that  $x(t)$  denotes the fraction of agents using action 1 in the population at time  $t$ . We consider the additive noise case as in Example 5. We assume in addition that the density  $g$  of  $\gamma$  is strictly positive on its support. As noted, Assumptions L and U are satisfied in this setting.

We consider a supermodular game, where  $\Delta u(x) := u(1, x) - u(0, x)$  is strictly increasing. We also assume  $\int_0^1 \Delta u(x) dx \neq 0$  to rule out a knife-edge case. If  $\int_0^1 \Delta u(x) dx > 0$ , then action 1 is a ‘‘Laplacian action’’ (Morris and Shin, 2003), i.e., the best response against the uniform distribution over the population states, which generalizes the risk dominant action in  $2 \times 2$  games. The following result shows that there is a unique  $(\epsilon)$ -equilibrium in the mean-field model under small noise  $\eta$  when agents are sufficiently patient.<sup>17</sup>

**Proposition 4.** *Consider the binary-action supermodular game with additive noise. Then there exist  $\bar{r} > 0$ ,  $\bar{\eta} > 0$ , and  $\delta > 0$  such that the mean-field model admits a unique  $\delta$ -equilibrium for  $\eta \leq \bar{\eta}$  and  $r \leq \bar{r}$ . In this equilibrium, action 1 (resp. 0) is chosen at almost all  $z$  if  $\int_0^1 \Delta u(x) dx > 0$  (resp.  $\int_0^1 \Delta u(x) dx < 0$ ).*

Thus for any  $r \leq \bar{r}$  and  $\eta \leq \bar{\eta}$ ,  $\Sigma^{*,\delta}$  is a singleton as described in the proposition. Since Assumption U holds, Theorem 2 implies the following equilibrium uniqueness result for the finite-population model with large  $N$  and small  $\eta$ .

**Corollary 2.** *Consider the binary-action supermodular game with additive noise, and let  $\bar{r}$  and  $\bar{\eta}$  be as in Proposition 4. Then if  $\eta \leq \bar{\eta}$ , there exists  $\bar{N}_\eta$  such that  $\Gamma_N$  admits a unique equilibrium for  $N \geq \bar{N}_\eta$  and  $r \leq \bar{r}$ . In this equilibrium, action 1 (resp. 0) is chosen at almost all  $z$  if  $\int_0^1 \Delta u(x) dx > 0$  (resp.  $\int_0^1 \Delta u(x) dx < 0$ ).*

The results show that, for sufficiently patient agents, the equilibrium becomes unique as the noise level  $\gamma$  becomes small while the population size  $N$  becomes large. Observe that it is crucial that  $N$  grows sufficiently fast relative to how  $\gamma$  shrinks.<sup>18</sup> In the unique equilibrium, agents always choose the Laplacian action, and the population state converges to the corresponding point-mass distribution. This strategy is also an equilibrium in a continuum-population model with perfect state observation, i.e.,  $N = \infty$  and  $\eta = 0$ . However, as observed in the literature (Matsuyama, 1991; Matsui and Matsuyama, 1995), such a model also admits many other equilibria, in which the non-Laplacian action is chosen at some population states. Our results show that such multiplicity disappears under appropriate choices of  $N \approx \infty$  and  $\eta \approx 0$ .

The equilibrium uniqueness reflects the intuition that the imperfectness of state observation, even small, can make it difficult for agents to coordinate behavior. In fact, interior threshold strategies are not supported in equilibrium. To see this, suppose that  $\int_0^1 \Delta u(x) dx > 0$  (so that action 1 is the Laplacian action), and consider the mean-field model with  $\eta, r \approx 0$ . Suppose that agents employ a threshold strategy that chooses action 1 if and only if the observed signal is higher than  $z^* \in (0, 1)$ . It follows that there is some value  $x^* \approx z^*$  such that the state converges to 1 as  $t \rightarrow \infty$  if the current state is higher than  $x^*$ , while it converges to 0 if the current state is lower than  $x^*$ . In the presence of observation noise  $\eta > 0$ , an incoming agent does not know which scenario prevails when she observes a signal (very close to)  $z^*$ . The key technical step of the proof is to show that such an agent’s interim belief approximately takes the following form: with probability  $1 - z^*$  (resp.  $z^*$ ), the current state is higher than  $x^*$  (resp. lower than  $x^*$ ) and all the future incoming agents choose action 1 (resp. action 0). Under such a belief, her continuation payoff to choosing action  $a$  is approximately calculated by

$$(1 - z^*)(1 + r) \int_0^\infty e^{-(1+r)t} u(a, e^{-t} z^* + (1 - e^{-t})) dt + z^*(1 + r) \int_0^\infty e^{-(1+r)t} u(a, e^{-t} z^*) dt \approx \int_0^1 u(a, x) dx$$

as  $r \approx 0$ , so that the agent has a strict incentive to choose action 1 by  $\int_0^1 \Delta u(x) dx > 0$ . This observation implies that such a threshold strategy cannot be supported as an equilibrium. One can also verify that it is not an equilibrium to always choose action 0, because  $\int_0^1 \Delta u(x) dx > 0$  implies that it is suboptimal to choose 0 at high signal  $z$ . In the Appendix, we indeed show that the unique equilibrium is to always choose action 1.

The following simple examples illustrate the result in concrete economic contexts (stated in terms of the mean-field model):

**Example 6 (Market thickness).** Decentralized markets often exhibit strategic complementarities among agents’ market participations by creating market thickness. Consider the setting in which each new entrant makes a binary irreversible choice, where action 1 is the choice to enter the market, and action 0 is not to enter. Thus  $x(t) \in [0, 1]$  represents the size of the market at  $t$ . If the agent chooses not to enter, she receives an outside option that yields a constant flow payoff  $\underline{w} > 0$ . Consider a random-match market as in Diamond (1982) and Diamond and Fudenberg (1989), in which at each time  $t$ , each agent in the market randomly meets another with a Poisson intensity  $\theta(x(t))$  and obtains a payoff  $\overline{w} > 0$  from trade, where  $\theta : [0, 1] \rightarrow \mathbb{R}_+$  is an increasing function representing the efficiency of matching technology. Thus the (expected) flow payoff of the market participation is given by  $u(1, x) = \theta(x)\overline{w}$ .<sup>19</sup> By Proposition 4, the equilibrium is unique under small noise  $\eta$  and discount rate  $r$ . In particular, all agents enter the market if  $\int_0^1 \theta(x) dx \cdot \frac{\overline{w}}{\underline{w}} > 1$ , i.e., the efficiency of matching technology and/or the relative size of gains from trade is large. If  $\int_0^1 \theta(x) dx \cdot \frac{\overline{w}}{\underline{w}} < 1$ , in contrast, the market breaks down as all agents leave.  $\diamond$

<sup>17</sup> The uniqueness of equilibrium is up to a set of signals that has Lebesgue measure zero.

<sup>18</sup> For any fixed  $N$ , the model is reduced to one with perfect state observation as  $\eta$  becomes sufficiently small, which admits multiple equilibria (Example 2).

<sup>19</sup> As pointed out by Morris and Shin (2012), an analogous market thickness effect can arise through mitigating adverse selection.

**Example 7 (Industrialization).** Strategic complementarities arise in the context of industrialization where agents choose their occupation sectors. Consider a small open economy that consists of self-employed agents. As in Matsuyama (1991), each new incoming agent makes an irreversible choice of her career. Each one can work in either the agriculture sector with constant returns to scale (sector 0) or the industrial sector with increasing returns (sector 1). All the prices are fixed (as determined at the global market). An agent in sector 0 produces one unit of output, independent of the sector configuration  $x(t)$ . On the other hand, an agent in sector 1 produces an amount of good that is increasing in the industry sector size  $x(t)$ . Under a standard specification of technology and preferences we obtain that  $\Delta u(x)$  is increasing in  $x$ , capturing strategic complementarity in industrialization.<sup>20</sup> Under small noise  $\eta$  and discount factor  $r$ , the proposition suggests that whether industrialization occurs or not can be uniquely determined as a function of economic primitives.  $\diamond$

Our equilibrium selection result in this section can be contrasted with those of the stochastic evolution models of Kandori et al. (1993) and Young (1993) for binary-action coordination games. In the latter, mutation-driven models, the long-run outcome is solely determined by the relative size of the basins of attraction (i.e., the best response regions), whereas the Laplacian outcome takes into account the magnitude of payoff differences (i.e., the potential). Furthermore, while those models entail long waiting times (see, e.g., Ellison, 1993), convergence to the Laplacian action in our model is fast, even when the population size is large.

## 5. Conclusion and discussion

This paper developed a simple framework to establish mean-field approximation results with forward-looking population dynamics. We introduced imperfectness in state observation to obtain continuity (in several forms) of the equilibrium behavior at the large population limit. Relative to the literature on perfect foresight dynamics (e.g., Matsuyama, 1991; Matsui and Matsuyama, 1995) and evolutionary dynamics (in particular, Benaïm and Weibull, 2003), first, our results provide a finite-population foundation to the former, which presumes the mean-field dynamics of a continuum population. Second, in contrast to the latter, mean-field approximation in our paper is uniform across all equilibria, which is essential because of potential equilibrium multiplicity due to agents' forward-looking incentives.

Below we discuss possible alternative modeling choices and some other issues of interest relative to this literature.

### 5.1. Observation noise versus payoff shocks

A key assumption of this paper is the presence of observation noise. This ensures the aggregate strategy  $E\sigma$  to be uniformly Lipschitz continuous in  $x$  (Lemma 1), which allows us to establish mean-field approximation uniformly across strategies (Lemma 2). In contrast, the previous literature on mean-field approximation with myopic agents often assumed *payoff shocks*, where agents' utilities are given by  $u(a, x) + \epsilon_a$  with i.i.d. random shocks  $(\epsilon_a)_{a \in A}$  that admit a continuous density (e.g., Fudenberg and Kreps, 1993; Hofbauer and Sandholm, 2002, 2007). With myopic agents, introducing such payoff shocks instead of observation noise ensures the aggregate strategy to be continuous. However, this argument would not work for forward-looking agents under perfect state observation. For instance, the strategy constructed in the perfect observation benchmark of Example 2 will continue to be an equilibrium in this case as long as the support of the payoff shocks is sufficiently small, and thus the continuity property as in Lemma 1 will fail to hold.

### 5.2. Prior beliefs

We assumed equilibrium expectations about the play of the future generations, while assuming exogenous inferences about the past play; in particular, agents' prior beliefs about the current population state are exogenously given. An alternative formulation would be to assume that the prior beliefs are formed in equilibrium. Maintaining the assumption that agents do not observe the calendar time, one may assume a prior over the calendar time, by which agents form beliefs on the current state from the equilibrium strategy. Our current formulation would correspond to the limiting case where the prior concentrates on time 0. Even with this alternative version, the results as well as their proofs in Sections 3.2–3.3 would be intact, as the details of agents' beliefs are irrelevant in the proofs. For the uniqueness result in Section 4, on the other hand, one would need to rule out priors that assign large weights on large  $t$ 's. If alternatively agents are allowed to observe the calendar time, the strategy of the agents and hence the aggregate dynamics would become non-stationary. It is left for future research to study such a non-stationary model and establish analogues of our approximation result and others.

### 5.3. Long-run approximation

Lemma 2 considers mean-field approximation of the state process as  $N \rightarrow \infty$  over any compact interval  $[0, T]$ , which is sufficient for analyzing agents' incentives in our model (due to time discounting). A different quantification of interest would be to consider the long-run empirical distribution of the state process as  $T \rightarrow \infty$  before taking the large population limit  $N \rightarrow \infty$ . While we do not pursue this direction, results along this line have been obtained by Benaïm (1998) and Benaïm and Weibull (2003), and we expect similar results would hold given our Lemma 1.

<sup>20</sup> See for example Matsuyama (1992b) for a micro foundation of this condition, and Oyama (2009) for a similar model in the context of spatial agglomeration.

**CRedit authorship contribution statement**

**Ryota Iijima:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Daisuke Oyama:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Appendix A. Proofs**

*A.1. Preliminaries*

**Lemma A.1.** For  $\sigma_n \in \Sigma$ ,  $n = 0, 1, \dots$ , if for all  $x \in \Delta_N$ ,  $(E\sigma_n)(x) \rightarrow (E\sigma_0)(x)$  as  $n \rightarrow \infty$ , then  $V_N(\cdot, \sigma_n) \rightarrow V_N(\cdot, \sigma_0)$  as  $n \rightarrow \infty$ .

**Proof.** Let  $T : (\mathbb{R}^{|A|})^{|\Delta_N|} \times \Sigma \rightarrow (\mathbb{R}^{|A|})^{|\Delta_N|}$  be the operator defined by

$$T(V, \sigma)(a, x) = \frac{1}{N+r} \left( (1+r)u(a, x) + (N-1) \sum_{a', a'' \in A} (E\sigma)(a'|x)x_{a''}^{-q} V \left( a, x + \frac{e_{a'} - e_{a''}}{N} \right) \right)$$

for  $x$  with  $x_a \geq \frac{1}{N}$  and  $T(V, \sigma)(a, x) = 0$  for  $x$  with  $x_a = 0$ .

For each  $n = 0, 1, \dots$ , the function  $V_N(\cdot, \sigma_n) \in (\mathbb{R}^{|A|})^{|\Delta_N|}$  is a fixed point of the operator  $V \mapsto T(V, \sigma_n)$ , which has a contraction factor  $\frac{N-1}{N+r} \in [0, 1)$  with respect to the sup norm  $\|\cdot\|$ . Then we have

$$\begin{aligned} & \|V_N(\cdot, \sigma_n) - V_N(\cdot, \sigma_0)\| \\ &= \|T(V_N(\cdot, \sigma_n), \sigma_n) - T(V_N(\cdot, \sigma_0), \sigma_0)\| \\ &\leq \|T(V_N(\cdot, \sigma_n), \sigma_n) - T(V_N(\cdot, \sigma_0), \sigma_n)\| + \|T(V_N(\cdot, \sigma_0), \sigma_n) - T(V_N(\cdot, \sigma_0), \sigma_0)\| \\ &\leq \frac{N-1}{N+r} \|V_N(\cdot, \sigma_n) - V_N(\cdot, \sigma_0)\| + \|T(V_N(\cdot, \sigma_0), \sigma_n) - T(V_N(\cdot, \sigma_0), \sigma_0)\|, \end{aligned}$$

and thus

$$\|V_N(\cdot, \sigma_n) - V_N(\cdot, \sigma_0)\| \leq \frac{N+r}{1+r} \|T(V_N(\cdot, \sigma_0), \sigma_n) - T(V_N(\cdot, \sigma_0), \sigma_0)\|.$$

Let  $n \rightarrow \infty$ . Then the right hand side of the above inequality converges to zero by the convergence  $(E\sigma_n)(x) \rightarrow (E\sigma_0)(x)$  for each  $x \in \Delta_N$ . Hence,  $\|V_N(\cdot, \sigma_n) - V_N(\cdot, \sigma_0)\| \rightarrow 0$  as desired.  $\square$

**Lemma A.2.** For any  $\epsilon > 0$  and  $T > 0$ , there exists  $\xi > 0$  such that

$$\max_{t \in [0, T]} \|\phi(t, x, \sigma) - \phi(t, x', \sigma')\| \leq \epsilon$$

holds for all  $\sigma, \sigma' \in \Sigma$  and  $x, x' \in \Delta$  with  $\|E\sigma - E\sigma'\| \leq \xi$  and  $\|x - x'\| \leq \xi$ .

**Proof.** Since  $\|F^\sigma(x) - F^\sigma(x')\| \leq \|(E\sigma)(x) - (E\sigma)(x')\| + \|x - x'\|$ , it follows from Lemma 1 that  $F^\sigma$  is Lipschitz continuous with a Lipschitz coefficient  $L' = L + 1$  uniform in  $\sigma$ . We therefore have

$$\begin{aligned} & \|\phi(t, x, \sigma) - \phi(t, x', \sigma')\| \\ &\leq \|x - x'\| + \int_0^t \|F^\sigma(\phi(s, x, \sigma)) - F^{\sigma'}(\phi(s, x', \sigma'))\| ds \\ &\leq \|x - x'\| + \int_0^t \left( L' \|\phi(s, x, \sigma) - \phi(s, x', \sigma')\| + \|F^\sigma - F^{\sigma'}\| \right) ds. \end{aligned}$$

From the Gronwall inequality, it follows that for any  $T$ ,

$$\|\phi(t, x, \sigma) - \phi(t, x', \sigma')\| \leq (\|x - x'\| + t\|F^\sigma - F^{\sigma'}\|)e^{L'T}$$

for all  $t \in [0, T]$ . Since  $\|F^\sigma - F^{\sigma'}\| \leq \|E\sigma - E\sigma'\|$ , given  $\epsilon > 0$  we have the desired inequality with  $\xi = \frac{\epsilon}{(1+T)e^{L'T}}$ .  $\square$

**Lemma A.3.** For any  $\epsilon > 0$ , there exists  $\xi > 0$  such that for any discount rate  $r > 0$ ,

$$|V(a, x, \sigma) - V(a, x', \sigma')| \leq \epsilon$$

holds for all  $\sigma, \sigma' \in \Sigma$  and  $x, x' \in \Delta$  with  $\|E\sigma - E\sigma'\| \leq \xi$  and  $\|x - x'\| \leq \xi$ , and  $a \in A$ .

**Proof.** Fix any  $\epsilon > 0$ . Let  $M = \sup_{x \in \Delta} \|u(x)\|$ , and take a  $T > 0$  such that  $2Me^{-T} \leq \frac{\epsilon}{2}$ , so that  $(1+r) \int_T^\infty e^{-(1+r)t} 2M dt \leq \frac{\epsilon}{2}$  for all  $r > 0$ . Let  $L_u$  denote the Lipschitz coefficient of  $u$ . By Lemma A.2, there exists  $\xi > 0$  such that

$$\max_{t \in [0, T]} \|\phi(t, x, \sigma) - \phi(t, x', \sigma')\| \leq \frac{\epsilon}{2L_u},$$

and hence,

$$\max_{t \in [0, T]} |u(a, \phi(t, x, \sigma)) - u(a, \phi(t, x', \sigma'))| \leq \frac{\epsilon}{2}$$

for all  $\sigma, \sigma' \in \Sigma$  and  $x, x' \in \Delta$  with  $\sup_{\bar{x}} \|E\sigma(\bar{x}) - E\sigma'(\bar{x})\| \leq \xi$  and  $\|x - x'\| \leq \xi$ , and  $a \in A$ . Then if  $\sup_{\bar{x}} \|E\sigma(\bar{x}) - E\sigma'(\bar{x})\| \leq \xi$  and  $\|x - x'\| \leq \xi$ , then for any  $r > 0$ , we have

$$\begin{aligned} |V(a, x, \sigma) - V(a, x', \sigma')| &\leq (1+r) \int_0^T e^{-(1+r)t} |u(a, \phi(t, x, \sigma)) - u(a, \phi(t, x', \sigma'))| dt \\ &\quad + (1+r) \int_T^\infty e^{-(1+r)t} 2M dt \leq \epsilon \end{aligned}$$

for all  $a \in A$ , as claimed.  $\square$

**Lemma A.4.** For any  $\epsilon > 0$  and  $z \in Z$ , there exists  $\bar{N}$  such that for any discount rate  $r > 0$ ,

$$\left| \int_{\Delta_N} V(a, x, \sigma) d\nu_{N,z}(x) - \int_{\Delta} V(a, x, \sigma) d\nu_z(x) \right| \leq \epsilon$$

holds for all  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ , and  $a \in A$ .

**Proof.** Within this proof, we make explicit the dependence of  $V$  on the discount rate  $r$  by writing  $V^r(x, \sigma)$  for  $V(x, \sigma)$ .

Fix any  $\epsilon > 0$ . The set  $\{V^r(\cdot, \sigma)\}_{r>0, \sigma \in \Sigma}$  of continuous functions on  $A \times \Delta$  is uniformly bounded and, by Lemma A.3, equicontinuous, and hence, is totally bounded in the sup norm by the Ascoli-Arzelà theorem. Therefore there is an  $\frac{\epsilon}{3}$ -net of  $\{V^r(\cdot, \sigma)\}_{r>0, \sigma \in \Sigma}$ , denoted by  $\{V^{rk}(\cdot, \sigma_k)\}_{k=1}^K$ , with finite  $K$ . Fix any  $z \in Z$ . Since  $\nu_{N,z} \rightarrow \nu_z$  weakly as  $N \rightarrow \infty$ , there exists  $\bar{N}$  such that

$$\left| \int_{\Delta_N} V^{rk}(a, x, \sigma_k) d\nu_{N,z}(x) - \int_{\Delta} V^{rk}(a, x, \sigma_k) d\nu_z(x) \right| \leq \frac{\epsilon}{3}$$

for all  $N \geq \bar{N}$ ,  $k = 1, \dots, K$ , and  $a \in A$ .

Fix any  $r > 0$ ,  $\sigma \in \Sigma$ , and let  $N \geq \bar{N}$ . Then we have, for some  $k$ ,

$$\begin{aligned} &\left| \int_{\Delta_N} V^r(a, x, \sigma) d\nu_{N,z}(x) - \int_{\Delta} V^r(a, x, \sigma) d\nu_z(x) \right| \\ &\leq \left| \int_{\Delta_N} V^r(a, x, \sigma) d\nu_{N,z}(x) - \int_{\Delta_N} V^{rk}(a, x, \sigma_k) d\nu_{N,z}(x) \right| \\ &\quad + \left| \int_{\Delta_N} V^{rk}(a, x, \sigma_k) d\nu_{N,z}(x) - \int_{\Delta} V^{rk}(a, x, \sigma_k) d\nu_z(x) \right| \\ &\quad + \left| \int_{\Delta} V^{rk}(a, x, \sigma_k) d\nu_z(x) - \int_{\Delta} V^r(a, x, \sigma) d\nu_z(x) \right| \leq \epsilon \end{aligned}$$

for all  $a \in A$ , as claimed.  $\square$

A.2. Proof of Propositions 1 and 2

We invoke an equilibrium existence theorem by Schmeidler (1973). In the following, we view the strategy space  $\Sigma$  as a subset of  $L_1(Z, \mathbb{R}^{|A|})$ , the set of (equivalence classes of) integrable functions from  $Z$  to  $\mathbb{R}^{|A|}$ , and endow it with the weak topology. Then  $\Sigma$  is a compact and convex subset of a locally convex linear topological space (see, e.g., Khan, 1985). For a sequence  $\{\sigma_n\} \subset \Sigma$  and  $\sigma_0 \in \Sigma$ ,  $\sigma_n \rightarrow \sigma_0$  weakly as  $n \rightarrow \infty$  if  $\int_Z \sigma_n(a|z)f(z)dz \rightarrow \int_Z \sigma_0(a|z)f(z)dz$  as  $n \rightarrow \infty$  for all  $a \in A$  and all  $f \in L_\infty(Z, \mathbb{R})$ , where  $L_\infty(Z, \mathbb{R})$  is the set of essentially bounded measurable functions.

For reference, we state the relevant result from Schmeidler (1973). For  $w_n : A \times Z \rightarrow \mathbb{R}$ ,  $n = 0, 1, \dots$ , write

$$B_n = \left\{ \sigma \in \Sigma \mid \text{for almost all } z \in Z : \sigma(a|z) > 0 \Rightarrow a \in \arg \max_{a' \in A} w_n(a', z) \right\}.$$

**Theorem A.1** (Schmeidler, 1973). *Suppose that*

- (a) *for all  $z \in Z$ ,  $w_n(\cdot, z) \rightarrow w_0(\cdot, z)$  as  $n \rightarrow \infty$ ; and*
- (b) *for all  $n = 0, 1, \dots$  and all  $a, a' \in A$ ,  $\{z \in Z \mid w_n(a, z) > w_n(a', z)\}$  is measurable.*

*Then for  $\sigma_n \in \Sigma$ ,  $n = 0, 1, \dots$ , if*

- *$\sigma_n \in B_n$  for all  $n \geq 1$ , and*
- *$\sigma_n \rightarrow \sigma_0$  weakly as  $n \rightarrow \infty$ ,*

*then  $\sigma_0 \in B_0$ .*

Note that condition (b) is satisfied if each  $w_n(a, \cdot)$  is measurable. We will use the following lemmas.

**Lemma A.5.** *For  $\sigma_n \in \Sigma$ ,  $n = 0, 1, \dots$ , if  $\sigma_n \rightarrow \sigma_0$  weakly as  $n \rightarrow \infty$ , then  $E\sigma_n \rightarrow E\sigma_0$  uniformly.*

**Proof.** The weak convergence  $\sigma_n \rightarrow \sigma_0$  implies that for each  $x \in \Delta$ ,  $(E\sigma_n)(x) \rightarrow (E\sigma_0)(x)$  (by the boundedness of the density function  $f_x$  for the case where  $Z$  has positive Lebesgue measure). From the compactness of  $\Delta$  and the uniform Lipschitz continuity of  $\{E\sigma_n\}$  by Lemma 1, it follows that  $E\sigma_n \rightarrow E\sigma_0$  uniformly.  $\square$

**Lemma A.6.** *For  $\sigma_n \in \Sigma$ ,  $n = 0, 1, \dots$ , if  $\sigma_n \rightarrow \sigma_0$  weakly as  $n \rightarrow \infty$ , then for each  $N$ ,  $z \in Z$ , and  $a \in A$ ,  $W_N(a, z, \sigma_n) \rightarrow W_N(a, z, \sigma_0)$  and  $W(a, z, \sigma_n) \rightarrow W(a, z, \sigma_0)$  hold as  $n \rightarrow \infty$ .*

**Proof.** The weak convergence  $\sigma_n \rightarrow \sigma_0$  implies that  $E\sigma_n \rightarrow E\sigma_0$  uniformly by Lemma A.5. Thus by Lemma A.1, we have  $V_N(\cdot, \sigma_n) \rightarrow V_N(\cdot, \sigma_0)$ , and hence, for each  $z \in Z$ ,  $W_N(\cdot, z, \sigma_n) \rightarrow W_N(\cdot, z, \sigma_0)$ . The convergence of  $W$  also follows since by Lemma A.3,  $V(\cdot, \sigma_n) \rightarrow V(\cdot, \sigma_0)$  as  $E\sigma_n \rightarrow E\sigma_0$  uniformly.  $\square$

Now we are ready to prove our existence theorems.

**Proof of Propositions 1 and 2.** We only prove Proposition 1. The proof of Proposition 2 is analogous with  $W$  in place of  $W_N$ .

Define the correspondence  $\beta : \Sigma \rightarrow \Sigma$  by

$$\beta(\sigma) = \left\{ \sigma' \in \Sigma \mid \text{for almost all } z \in Z : \sigma'(a|z) > 0 \Rightarrow a \in \arg \max_{a' \in A} W_N(a', z, \sigma) \right\}.$$

We want to show that  $\beta$  has a fixed point, which is an equilibrium of the game  $\Gamma_N$ , by applying the Kakutani-Fan-Glicksberg fixed point theorem.

The convexity of  $\beta(\sigma)$  is clear by construction. For the nonemptiness of  $\beta(\sigma)$ , order the actions as  $a_1, \dots, a_{|A|}$ , and for each  $i$ , let  $Z_i = \{z \in Z \mid z \notin \bigcup_{j=1}^{i-1} Z_j \text{ and } W_N(a_i, z, \sigma) \geq W_N(a', z, \sigma) \text{ for all } a' \in A\}$ . Since  $W_N$  is measurable in  $z$ , these sets are measurable. Then the strategy  $\sigma' \in \Sigma$  defined by  $\sigma'(a_i|z) = 1$  if and only if  $z \in Z_i$  is in  $\beta(\sigma)$ . To show the closedness of the graph of  $\beta$ , it suffices to consider sequences, instead of nets, as described in Khan (1985). So let  $\{(\sigma_n, \sigma'_n)\}$  be a sequence converging to  $(\sigma, \sigma')$  where  $\sigma'_n \in \beta(\sigma_n)$ . To apply Theorem A.1, let  $w_n(\cdot) = W_N(\cdot, \sigma_n)$  and  $w_0(\cdot) = W_N(\cdot, \sigma)$ , and thus  $B_n = \beta(\sigma_n)$  and  $B_0 = \beta(\sigma)$ . Then conditions (a) and (b) in Theorem A.1 are satisfied by Lemma A.6 and the measurability of  $W_N$ , and hence, it follows from Theorem A.1 that  $\sigma' \in \beta(\sigma)$ . This implies that  $\beta$  has a closed graph. Therefore, by the Kakutani-Fan-Glicksberg fixed point theorem,  $\beta$  has a fixed point.  $\square$

A.3. Proof of Lemma 2

The proof is largely based on Benaïm and Weibull (2003) with two modifications. First, we consider a continuous-time Poisson clock setting for the finite-agent models rather than a sequence of discrete period processes. We follow the proof sketch provided by Benaïm and Weibull (2003, Appendix I), while filling some gaps. Second, and more importantly, our approximation result is required

to be uniform across the family of mean-field dynamics induced by all strategies  $\sigma$ , while one particular mean-field dynamics is fixed in Benaïm and Weibull (2003).

In the following we use  $\|\cdot\|_2$  to denote the  $L_2$  norm. Let  $\Gamma(\kappa) = \frac{\sqrt{2\kappa}}{\log(1+\sqrt{2^{-1}\kappa})}$  for each  $\kappa > 0$ . For each  $N, \sigma \in \Sigma, x_0 \in \Delta_N, \kappa > 0$ , and  $\theta \in \mathbb{R}^{|A|}$ , we define the process

$$Z_N(t, \sigma) = \exp \left( \theta \cdot \left( x_N(t, \sigma) - \int_0^t F^\sigma(x_N(s, \sigma)) ds - x_0 \right) - \frac{t\Gamma(\kappa)\|\theta\|_2^2}{2N} \right). \tag{A.1}$$

We begin with the following preliminary lemma.

**Lemma A.7.** Take any  $N$  and  $\kappa > 0$  and any  $\theta \in \mathbb{R}^{|A|}$  such that  $\|\theta\|_2 = \frac{\kappa N}{\Gamma(\kappa)}$ . Then the process  $Z_N(t, \sigma)$  defined in (A.1) is a supermartingale for any  $\sigma \in \Sigma$  and  $x_0 \in \Delta_N$ .

**Proof.** For each  $\sigma \in \Sigma, x \in \Delta$ , and  $a, a' \in A$ , let  $P_x^\sigma(a, a') := (E\sigma)(a|x)_{x_{a'}}$  denote the probability that an agent with action  $a'$  is replaced by a new one with action  $a$  under strategy  $\sigma$ . Observe that  $\sum_{a, a' \in A} P_x^\sigma(a, a')(e_a - e_{a'}) = (E\sigma)(x) - x = F^\sigma(x)$ .

For each  $N$  and  $\sigma \in \Sigma$ , let  $L_N^\sigma$  denote the infinitesimal generator:

$$\begin{aligned} L_N^\sigma(f)(x_0) &= \lim_{t \rightarrow 0} \frac{\mathbb{E}[f(x_N(t, \sigma)) - f(x_0) | x_N(0) = x_0]}{t} \\ &= N \sum_{a, a' \in A} \left( f \left( x_0 + \frac{e_a - e_{a'}}{N} \right) - f(x_0) \right) P_{x_0}^\sigma(a, a') \end{aligned}$$

for any  $x_0 \in \Delta_N$  and any continuous function  $f$  defined on  $\Delta$ .

Let  $f(x) = \exp(\theta \cdot x)$  and  $g(w) = \exp(w) - w - 1$ . Then we have<sup>21</sup>

$$\begin{aligned} \frac{L_N^\sigma(f)(x)}{f(x)} &= N \left( \sum_{a, a' \in A} P_x^\sigma(e_a - e_{a'}) \exp \left( \frac{\theta \cdot (e_a - e_{a'})}{N} \right) - 1 \right) \\ &= F^\sigma(x) \cdot \theta + N \sum_{a, a' \in A} g \left( \frac{\theta \cdot (e_a - e_{a'})}{N} \right) P_x^\sigma(a, a') \end{aligned} \tag{A.2}$$

using  $F^\sigma(x) = \sum_{a, a' \in A} P_x^\sigma(a, a')(e_a - e_{a'})$ .

Define  $h(w) = \frac{\Gamma(\kappa)}{4} w^2$ . Then  $g'(w) - h'(w) = 0$  at  $w = 0, \frac{\sqrt{2\kappa}}{\Gamma(\kappa)}$  (by construction of  $\Gamma(\cdot)$ ). Since  $g' - h'$  is convex, this implies  $g'(w) - h'(w) \leq 0$  holds for all  $w \in [0, \frac{\sqrt{2\kappa}}{\Gamma(\kappa)}]$ . As  $g(0) = h(0) = 0$ , we have  $g(w) \leq h(w)$  for all  $w \in [0, \frac{\sqrt{2\kappa}}{\Gamma(\kappa)}]$ . Thus, for each  $\ell \in \mathcal{E}$ ,

$$Ng \left( \frac{\theta \cdot \ell}{N} \right) \leq Ng \left( \frac{\|\theta\|_2 \sqrt{2}}{N} \right) \leq Nh \left( \frac{\|\theta\|_2 \sqrt{2}}{N} \right) = \frac{\Gamma(\kappa)\|\theta\|_2^2}{2N}$$

holds, where the first inequality follows from the Cauchy-Schwarz inequality and the fact that  $g$  is increasing, and the second inequality uses  $\|\theta\|_2 = \frac{\kappa N}{\Gamma(\kappa)}$ . By (A.2), this ensures

$$\frac{L_N^\sigma(f)(x)}{f(x)} - F^\sigma(x) \cdot \theta \leq \frac{\Gamma(\kappa)\|\theta\|_2^2}{2N}.$$

Thus, we have

$$\begin{aligned} &\lim_{\tau \rightarrow 0} \frac{\mathbb{E}[Z_N(t + \tau, \sigma) - Z_N(t, \sigma) | x_N(0) = x_0]}{\tau} \\ &= Z_N(t, \sigma) \left( \frac{L_N^\sigma(f)(x_N(t, \sigma))}{f(x_N(t, \sigma))} - F^\sigma(x_N(t, \sigma)) \cdot \theta - \frac{\Gamma(\kappa)\|\theta\|_2^2}{2N} \right) \leq 0, \end{aligned}$$

and hence,  $Z_N(t, \sigma)$  is a supermartingale.  $\square$

**Proof of Lemma 2.** Fix any  $\epsilon > 0$  and  $T > 0$ . Let  $L' = L + 1$  denote the Lipschitz coefficient of  $F^\sigma$ , which is independent of  $\sigma$  by Lemma 1. Define

<sup>21</sup> The corresponding functions (incorrectly) stated in Benaïm and Weibull (2003, Appendix I) are  $f(x) = \theta \cdot (x - x_0)$  and  $g(w) = \exp(w) - w + 1$  in our notation. Likewise the corresponding process  $Z_N$  is stated as  $Z_N(t, \sigma) = \exp \left( \theta \cdot \left( x_N(t, \sigma) - \int_0^t F^\sigma(x_N(s)) ds - x_0 \right) - \frac{t\Gamma(\kappa)\|\theta\|_2^2}{N} \right)$ .

$$\Psi_N(T, \sigma) = \max_{0 \leq t \leq T} \left\| x_N(t, \sigma) - x_0 - \int_0^t F^\sigma(x_N(\tau, \sigma)) d\tau \right\|.$$

Since, by definition,

$$\phi(t, \sigma) = x_0 + \int_0^t F^\sigma(\phi(\tau, \sigma)) d\tau$$

for each  $t$  (where we omit  $\phi$ 's dependence on  $x_0$ ), we have

$$\begin{aligned} \|x_N(t, \sigma) - \phi(t, \sigma)\| &= \left\| x_N(t, \sigma) - x_0 - \int_0^t F^\sigma(\phi(\tau)) d\tau \right\| \\ &\leq \left\| x_N(t, \sigma) - x_0 - \int_0^t F^\sigma(x_N(\tau, \sigma)) d\tau \right\| \\ &\quad + \left\| \int_0^t (F^\sigma(x_N(\tau, \sigma)) - F^\sigma(\phi(\tau, \sigma))) d\tau \right\| \\ &\leq \Psi_N(T, \sigma) + L' \int_0^t \|x_N(\tau, \sigma) - \phi(\tau, \sigma)\| d\tau. \end{aligned}$$

Then a version of the Gronwall inequality (Ethier and Kurtz, 2009, Theorem 5.1 in Appendix) implies that

$$\max_{0 \leq t \leq T} \|x_N(t, \sigma) - \phi(t, \sigma)\| \leq \Psi_N(T, \sigma) e^{L'T}.$$

Therefore, we have

$$\Pr \left[ \max_{0 \leq t \leq T} \|x_N(t, \sigma) - \phi(t, \sigma)\| \geq \epsilon \right] \leq \Pr \left[ \Psi_N(T, \sigma) \geq \epsilon e^{-L'T} \right].$$

To complete the proof, we find an exponential bound on the right hand side. Take any  $\xi, \beta > 0$ . For  $\kappa = \frac{\xi}{T}$  and any  $\theta \in \mathbb{R}^{|A|}$  with  $\|\theta\|_2 = \frac{\kappa N}{\Gamma}$ , where  $\Gamma = \Gamma(\frac{\xi}{T})$ , consider the process  $Z_N(t, \sigma)$  as defined in (A.1). Then

$$\begin{aligned} &\Pr \left[ \max_{0 \leq t \leq T} \theta \cdot (x_N(t, \sigma) - \int_0^t F^\sigma(x_N(s, \sigma)) ds - x_0) \geq \beta \right] \\ &\leq \Pr \left[ \max_{0 \leq t \leq T} Z_N(t, \sigma) \geq \exp \left( \beta - \frac{T\Gamma\|\theta\|_2^2}{2N} \right) \right] \\ &\leq \exp \left( \frac{T\Gamma\|\theta\|_2^2}{2N} - \beta \right), \end{aligned}$$

where the second inequality follows from Doob's supermartingale inequality with  $Z_N(0, \sigma) = 1$ , where  $Z_N(t, \sigma)$  is a supermartingale by Lemma A.7.

For each  $a \in A$ , applying the above inequality with setting  $\theta = \frac{e_a \beta}{\xi}$  or  $-\frac{e_a \beta}{\xi}$  and  $\beta = \frac{\xi^2 N}{\Gamma T}$  (which guarantees  $\|\theta\|_2 = \frac{\kappa N}{\Gamma(\kappa)}$ , where  $\kappa = \frac{\xi}{T}$ ), we have

$$\Pr \left[ \max_{0 \leq t \leq T} d \cdot (x_N(t, \sigma) - \int_0^t F^\sigma(x_N(s, \sigma)) ds - x_0) \geq \xi \right] \leq \exp \left( -\frac{\beta}{2} \right) = \exp \left( \frac{-\xi^2 N}{2\Gamma T} \right),$$

where  $d = e_a$  or  $-e_a$ . This implies

$$\Pr \left[ \Psi_N(T, \sigma) \geq \xi \right] \leq 2|A| \exp \left( \frac{-\xi^2 N}{2\Gamma T} \right).$$

Let  $\xi = \epsilon e^{-L'T}$ . Then we have

$$\Pr \left[ \Psi_N(T, \sigma) \geq \epsilon e^{-L'T} \right] \leq 2|A| \exp \left( \frac{-\epsilon^2 e^{-2L'T} N}{2\Gamma T} \right) = 2|A| e^{-2CN},$$

where  $C = \frac{\epsilon^2 e^{-2L'T}}{4\Gamma T}$  with  $\Gamma = \Gamma(\frac{\xi}{T})$ . Therefore, we have

$$\Pr \left[ \max_{0 \leq t \leq T} \|x_N(t, \sigma) - \phi(t, \sigma)\| \geq \epsilon \right] \leq e^{-CN}$$

for all  $N \geq \frac{1}{C} \log(2|A|)$ .  $\square$

**A.4. Proof of Lemma 3**

**Proof.** Fix any  $\epsilon > 0$ . Let  $M = \max_{x \in \Delta_N} \|u(x)\|$ , and take a  $T > 0$  such that  $2Me^{-T} \leq \frac{\epsilon}{2}$ , so that  $(1+r) \int_T^\infty e^{-(1+r)t} 2M dt \leq \frac{\epsilon}{2}$  for all  $r > 0$ . Let  $L_u$  denote the Lipschitz coefficient of  $u$ . By Lemma 2, there exists  $\bar{N}$  such that  $\mathbb{E}^x [\sup_{t \in [0, T]} \|x_{N-1}(t, x, \sigma) - \phi(t, x, \sigma)\|] \leq \frac{\epsilon}{2L_u} - \frac{1}{N}$  for all  $\sigma \in \Sigma$ ,  $N \geq \bar{N}$ , and  $x \in \Delta_N$ , where we write  $\mathbb{E}^x[\cdot]$  for  $\mathbb{E}[\cdot | x_N(0) = x]$ .

Now fix any  $r > 0$ , and take any  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $x \in \Delta_N$ , and  $a \in A$ . We note that the continuation value can be written as

$$V_N(a, x, \sigma) = (1+r) \mathbb{E}^x \left[ \int_0^\infty e^{-(1+r)t} u \left( a, \frac{e_a + (N-1)x_{N-1}(t, x, \sigma)}{N} \right) dt \right]. \tag{A.3}$$

Then we have

$$\begin{aligned} & |V_N(a, x, \sigma) - V(a, x, \sigma)| \\ & \leq (1+r) \mathbb{E}^x \left[ \int_0^T e^{-(1+r)t} \left| u \left( a, \frac{e_a + (N-1)x_{N-1}(t, x, \sigma)}{N} \right) - u(a, \phi(t, x, \sigma)) \right| dt \right] \\ & \quad + (1+r) \int_T^\infty e^{-(1+r)t} 2M dt \\ & \leq \mathbb{E}^x \left[ \sup_{t \in [0, T]} L_u \frac{1 + (N-1)\|x_{N-1}(t, x, \sigma) - \phi(t, x, \sigma)\|}{N} \right] + \frac{\epsilon}{2} \leq \epsilon, \end{aligned}$$

as claimed.  $\square$

Combined with Lemma A.3, Lemma 3 is strengthened as in the following, which will be used to prove Theorem 2.

**Lemma A.8.** For any  $\epsilon > 0$  there exist  $\bar{N}$  and  $\xi > 0$  such that for any discount rate  $r > 0$ ,

$$|V_N(a, x, \sigma) - V(a, x, \sigma')| \leq \epsilon$$

holds for all  $N \geq \bar{N}$ ,  $\sigma, \sigma' \in \Sigma$  with  $\|E\sigma - E\sigma'\| \leq \xi$ ,  $x \in \Delta_N$ , and  $a \in A$ .

**Proof.** Fix any  $\epsilon > 0$ . By Lemma 3, there exists  $\bar{N}$  such that for any  $r > 0$ ,

$$|V_N(a, x, \sigma) - V(a, x, \sigma)| \leq \frac{\epsilon}{2}$$

for all  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $x \in \Delta_N$ , and  $a \in A$ . By Lemma A.3, there exists  $\xi > 0$  such that for any  $r > 0$ ,

$$|V(a, x, \sigma) - V(a, x, \sigma')| \leq \frac{\epsilon}{2}$$

for all  $\sigma, \sigma' \in \Sigma$  with  $\|E\sigma - E\sigma'\| \leq \xi$ ,  $x \in \Delta_N$ , and  $a \in A$ . Therefore, if  $N \geq \bar{N}$  and  $\|E\sigma - E\sigma'\| \leq \xi$ , then for any  $r > 0$ , we have

$$|V_N(a, x, \sigma) - V(a, x, \sigma)| \leq |V_N(a, x, \sigma) - V(a, x, \sigma)| + |V(a, x, \sigma) - V(a, x, \sigma')| \leq \epsilon$$

for all  $x \in \Delta_N$  and  $a \in A$ , as claimed.  $\square$

**A.5. Proof of Theorem 1**

**Lemma A.9.** For any  $\epsilon > 0$ , there exists  $\bar{N}$  such that for any discount rate  $r > 0$ ,

$$\left| V_N \left( a, x + \frac{e_a - e_{a'}}{N}, \sigma \right) - V_N(a, x, \sigma) \right| \leq \epsilon$$

holds for all  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $x \in \Delta_N$ , and  $a, a' \in A$ .

**Proof.** Fix any  $\epsilon > 0$ . First, by Lemma 3, there exists  $N_1$  such that for any  $r > 0$ ,

$$|V_N(a, x, \sigma) - V(a, x, \sigma)| \leq \frac{\epsilon}{3}$$

for all  $N \geq N_1$ ,  $\sigma \in \Sigma$ ,  $x \in \Delta_N$ , and  $a \in A$ . Second, by Lemma A.3, there exists  $\xi > 0$  such that for any  $r > 0$ ,

$$|V(a, x, \sigma) - V(a, x', \sigma)| \leq \frac{\epsilon}{3}$$

for all  $\sigma \in \Sigma$ ,  $x, x' \in \Delta$  with  $\|x - x'\| \leq \xi$ , and  $a \in A$ . Let  $N_2 = \lceil \frac{1}{\xi} \rceil$ . Then, if  $N \geq N_2$ , then for any  $r > 0$ ,

$$\left| V\left(a, x + \frac{e_a - e_{a'}}{N}, \sigma\right) - V(a, x, \sigma) \right| \leq \frac{\epsilon}{3}$$

for all  $\sigma \in \Sigma$ ,  $x \in \Delta$ , and  $a, a' \in A$ . Let  $\bar{N} = \max\{N_1, N_2\}$ .

Now fix any  $r > 0$ , and take any  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $x \in \Delta_N$ , and  $a, a' \in A$ . Then we have

$$\begin{aligned} & \left| V_N\left(a, x + \frac{e_a - e_{a'}}{N}, \sigma\right) - V_N(a, x, \sigma) \right| \\ & \leq \left| V_N\left(a, x + \frac{e_a - e_{a'}}{N}, \sigma\right) - V\left(a, x + \frac{e_a - e_{a'}}{N}, \sigma\right) \right| \\ & \quad + \left| V\left(a, x + \frac{e_a - e_{a'}}{N}, \sigma\right) - V(a, x, \sigma) \right| + |V(a, x, \sigma) - V_N(a, x, \sigma)| \leq \epsilon, \end{aligned}$$

as claimed.  $\square$

**Lemma A.10.** For any  $\epsilon > 0$ , there exists  $\bar{N}$  such that for any discount rate  $r > 0$ ,

$$\left| W_N(a, z, \sigma) - \int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x) \right| \leq \epsilon$$

holds for all  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $z \in Z$ , and  $a \in A$ .

**Proof.** Fix any  $\epsilon > 0$ . By Lemma A.9, there exists  $\bar{N}$  such that for any  $r > 0$ ,

$$\left| V_N\left(a, x + \frac{e_a - e_{a'}}{N}, \sigma\right) - V_N(a, x, \sigma) \right| \leq \epsilon$$

holds for all  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $x \in \Delta_N$ , and  $a, a' \in A$ . Therefore, if  $N \geq \bar{N}$ , then for any  $r > 0$ , we have

$$\begin{aligned} & \left| W_N(a, z, \sigma) - \int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x) \right| \\ & \leq \int_{\Delta_N} \sum_{a' \in A} x_{a'} \left| V_N\left(a, x + \frac{e_a - e_{a'}}{N}, \sigma\right) - V_N(a, x, \sigma) \right| d\nu_{N,z}(x) \leq \epsilon \end{aligned}$$

for all  $\sigma \in \Sigma$ ,  $z \in Z$ , and  $a \in A$ , as claimed.  $\square$

**Proof of Theorem 1.** Fix any  $\epsilon > 0$ . By Lemma A.10, there exists  $\bar{N}$  such that for any  $r > 0$ ,

$$\left| W_N(a, z, \sigma) - \int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x) \right| \leq \frac{\epsilon}{2}$$

for all  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $z \in Z$ , and  $a \in A$ .

Fix any  $r > 0$ , and let  $N \geq \bar{N}$ . Take any  $\sigma \in \Sigma_N^*$ , and suppose that  $\sigma(a^*|z) > 0$ . For all  $a \in A$ , we have

$$\int_{\Delta_N} V_N(a^*, x, \sigma) d\nu_{N,z}(x) - \int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x) \geq W_N(a^*, z, \sigma) - W_N(a, z, \sigma) - \epsilon,$$

where by the optimality of  $\sigma$ ,  $W_N(a^*, z, \sigma) - W_N(a, z, \sigma) \geq 0$  for almost all  $z \in Z$ . This proves that any  $\sigma \in \Sigma_N^*$  satisfies the  $\epsilon$ -small agent property whenever  $N \geq \bar{N}$ .  $\square$

A.6. Proof of Corollary 1

**Proof.** Let  $a \in A$  be dominated. There exists  $\delta > 0$  such that  $\min_{x \in \Delta} u(\bar{a}, x) - u(a, x) > \delta$ . Since  $u$  is Lipschitz continuous, there is  $N_1$  such that  $\min_{x \in \Delta} u(\bar{a}, \frac{e_{\bar{a}} + (N-1)x}{N}) - u(a, \frac{e_a + (N-1)x}{N}) > \delta$  for all  $N \geq N_1$ . By Theorem 1, there exists  $N_2$  such that for any  $r > 0$  and any  $N \geq \bar{N}$ , any  $\sigma \in \Sigma_N^*$  satisfies the  $\delta$ -small agent property. Let  $\bar{N} = \max\{N_1, N_2\}$ .

Take any  $r > 0$ ,  $N \geq \bar{N}$ , and  $\sigma \in \Sigma_N^*$ . By (A.3) we have

$$\begin{aligned} & V_N(\bar{a}, x, \sigma) - V_N(a, x, \sigma) \\ &= (1+r) \mathbb{E}^x \left[ \int_0^\infty e^{-(1+r)s} \left( u\left(\bar{a}, \frac{e_{\bar{a}} + (N-1)x_{N-1}(s, x, \sigma)}{N}\right) - u\left(a, \frac{e_a + (N-1)x_{N-1}(s, x, \sigma)}{N}\right) \right) ds \right] > \delta \end{aligned}$$

for all  $x \in \Delta$ , and hence,

$$\int_{\Delta_N} V_N(\bar{a}, x, \sigma) d\nu_{N,z}(x) - \int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x) > \delta.$$

By the  $\delta$ -small agent property of  $\sigma$ , this implies that  $\sigma(a|z) = 0$  for almost all  $z \in Z$ .  $\square$

A.7. Proofs of Proposition 3 and Theorem 2

**Lemma A.11.** For any  $\epsilon > 0$ , there exists  $\xi > 0$  such that for any  $z \in Z$ , there exists  $\bar{N}$  such that for any discount rate  $r > 0$ ,

$$|W_N(a, z, \sigma') - W(a, z, \sigma)| \leq \epsilon$$

holds for all  $N \geq \bar{N}$ ,  $\sigma, \sigma' \in \Sigma$  with  $\|E\sigma' - E\sigma\| \leq \xi$ , and  $a \in A$ .

**Proof.** Fix any  $\epsilon > 0$ . By Lemma A.10, there exists  $N_1$  such that for any  $r > 0$ ,

$$\left| W_N(a, z, \sigma) - \int_{\Delta_N} V_N(a, x, \sigma) d\nu_{N,z}(x) \right| \leq \frac{\epsilon}{3}$$

holds for all  $N \geq N_1$ ,  $\sigma \in \Sigma$ ,  $z \in Z$ , and  $a \in A$ , while by Lemma A.8, there exists  $N_2$  and  $\xi > 0$  such that for any  $r > 0$ ,

$$|V_N(a, x, \sigma') - V(a, x, \sigma)| \leq \frac{\epsilon}{3}$$

holds for any  $N \geq N_2$ ,  $\sigma, \sigma' \in \Sigma$  with  $\|E\sigma' - E\sigma\| \leq \xi$ ,  $x \in \Delta_N$ , and  $a \in A$ .

Fix any  $z \in Z$ . By Lemma A.4, there exists  $N_3$  such that for any  $r > 0$ ,

$$\left| \int_{\Delta_N} V(a, x, \sigma) d\nu_{N,z}(x) - \int_{\Delta} V(a, x, \sigma) d\nu_z(x) \right| \leq \frac{\epsilon}{3}$$

for all  $N \geq N_3$ ,  $\sigma \in \Sigma$ , and  $a \in A$ . Let  $\bar{N} = \max\{N_1, N_2, N_3\}$ .

Fix any  $r > 0$ , and let  $N \geq \bar{N}$ . Take any  $\sigma, \sigma' \in \Sigma$  with  $\|E\sigma' - E\sigma\| \leq \xi$ . Then we have

$$\begin{aligned} |W_N(a, z, \sigma') - W(a, z, \sigma)| &\leq \left| W_N(a, z, \sigma') - \int_{\Delta_N} V_N(a, x, \sigma') d\nu_{N,z}(x) \right| \\ &\quad + \int_{\Delta_N} |V_N(a, x, \sigma') - V(a, x, \sigma)| d\nu_{N,z}(x) \\ &\quad + \left| \int_{\Delta_N} V(a, x, \sigma) d\nu_{N,z}(x) - \int_{\Delta} V(a, x, \sigma) d\nu_z(x) \right| \leq \epsilon, \end{aligned}$$

as claimed.  $\square$

**Lemma A.12.** For any sequence  $(\sigma_N)_{N \in \mathbb{N}}$  in  $\Sigma$ , if  $E\sigma_N \rightarrow E\sigma$  uniformly as  $N \rightarrow \infty$ , then for any  $z \in Z$  and  $a \in A$ ,  $W_N(a, z, \sigma_N) \rightarrow W(a, z, \sigma)$  as  $N \rightarrow \infty$ .

**Proof.** Suppose that  $E\sigma_N \rightarrow E\sigma$  uniformly as  $N \rightarrow \infty$ . Fix any  $\epsilon > 0$ . Let  $\xi > 0$  be as in Lemma A.11. Given  $z \in Z$ , let  $\bar{N}$  be as in Lemma A.11. Let  $\bar{N}' \geq \bar{N}$  be such that  $\|E\sigma_N - E\sigma\| \leq \xi$  for all  $N \geq \bar{N}'$ . Then we have  $|W_N(a, z, \sigma_N) - W(a, z, \sigma)| \leq \epsilon$  for all  $N \geq \bar{N}'$ .  $\square$

**Proposition A.1.** For any sequence  $(\sigma_N)_{N \in \mathbb{N}}$  where  $\sigma_N \in \Sigma_N^*$  for each  $N$ , there exist a subsequence  $(\sigma_{N_k})_{k \in \mathbb{N}}$  of  $(\sigma_N)_{N \in \mathbb{N}}$  and  $\sigma \in \Sigma^*$  such that  $\sigma_{N_k} \rightarrow \sigma$  weakly as  $k \rightarrow \infty$ .

**Proof.** Take any sequence  $(\sigma_N)_{N \in \mathbb{N}}$  where  $\sigma_N \in \Sigma_N^*$  for each  $N$ . By the weak (sequential) compactness of  $\Sigma$ ,  $(\sigma_N)_{N \in \mathbb{N}}$  has a weakly convergent subsequence, again denoted  $(\sigma_N)_{N \in \mathbb{N}}$ , with a limit  $\sigma \in \Sigma$ . We want to show that  $\sigma$  is an equilibrium of  $\Gamma$ .

To apply Theorem A.1, let  $w_N = W_N(\cdot, \sigma_N)$  and  $w_0 = W(\cdot, \sigma)$ . By Lemma A.5, the weak convergence of  $\sigma_N \rightarrow \sigma$  implies the uniform convergence of  $E\sigma_N \rightarrow E\sigma$ . Therefore, by Lemma A.12, for each  $z \in Z$  and  $a \in A$ ,  $W_N(a, z, \sigma_N) \rightarrow W(a, z, \sigma)$  as  $N \rightarrow \infty$ ; thus condition (a) in Theorem A.1 is satisfied. Condition (b) holds by the measurability of  $W_N$  and  $W$ . Hence, it follows from Theorem A.1 that for almost all  $z \in Z$ ,  $a \in \arg \max_{a' \in A} W(a', z, \sigma)$  whenever  $\sigma(a|z) > 0$ , i.e.,  $\sigma$  is an equilibrium of  $\Gamma$ .  $\square$

**Proof of Proposition 3.** Follows from Proposition A.1 along with Lemmas A.2 and A.5.  $\square$

**Proof of Theorem 2.** Fix any  $\epsilon > 0$ . Under Assumption U, inspecting their proofs shows that the  $\bar{N}$  in Lemma A.4 and hence that in Lemma A.11 can in fact be taken uniformly over all  $z \in Z$ . Thus, there exists  $\bar{N}$  such that for any  $r > 0$ ,

$$|W_N(a, z, \sigma) - W(a, z, \sigma)| \leq \frac{\epsilon}{2}$$

for all  $N \geq \bar{N}$ ,  $\sigma \in \Sigma$ ,  $z \in Z$ , and  $a \in A$ .

Fix any  $r > 0$ , and let  $N \geq \bar{N}$ . Take any  $\sigma \in \Sigma_N^*$ , and suppose that  $z \in Z$  and  $\sigma(a^*|z) > 0$ . Then for all  $a \in A$ , we have

$$W(a^*, z, \sigma) - W(a, z, \sigma) \geq W_N(a^*, z, \sigma) - W_N(a, z, \sigma) - \epsilon,$$

where by the optimality of  $\sigma$  in  $\Gamma_N$ ,  $W_N(a^*, z, \sigma) - W_N(a, z, \sigma) \geq 0$  for almost all  $z \in Z$ . Hence,  $\sigma \in \Sigma^{*,\epsilon}$ .

Similarly, take any  $\sigma \in \Sigma^*$ , and suppose that  $z \in Z$  and  $\sigma(a^*|z) > 0$ . Then for all  $a \in A$ , we have

$$W_N(a^*, z, \sigma) - W_N(a, z, \sigma) \geq W(a^*, z, \sigma) - W(a, z, \sigma) - \epsilon,$$

where by the optimality of  $\sigma$  in  $\Gamma$ ,  $W(a^*, z, \sigma) - W(a, z, \sigma) \geq 0$  for almost all  $z \in Z$ . Hence,  $\sigma \in \Sigma_N^{*,\epsilon}$ .  $\square$

#### A.8. Proof of Proposition 4

For each  $\sigma \in \Sigma$ , denote

$$\begin{aligned} \Delta V(x, \sigma) &= (1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\phi(t, x, \sigma)) dt, \\ \Delta W(z, \sigma) &= \int_{[0,1]} \Delta V(x, \sigma) d\nu_{z,\eta}(x), \end{aligned}$$

where  $\phi(\cdot, x, \sigma)$  is the mean-field path under  $\sigma$  with the initial state  $x$ . We only consider the case of  $\int_0^1 \Delta u(x) dx > 0$ ; the other case follows from the symmetric argument. We let

$$\bar{\delta} = \frac{1}{2} \int_0^1 \Delta u(x) dx$$

and show that for sufficiently small  $r$  and  $\eta$ , there is a unique mean-field  $\bar{\delta}$ -equilibrium, which consists in always playing action 1. Let  $\bar{\alpha} > 0$  be such that

$$\int_0^1 \Delta u((1 - 2\bar{\alpha})y) dy > \bar{\delta}.$$

In the following, we assume that  $\text{supp } G = [-1, 1]$  to simplify the notation; it is straightforward to consider more general cases. Let  $\bar{g} := \max_{\gamma \in [-1, 1]} g(\gamma)$  and  $\underline{g} := \min_{\gamma \in [-1, 1]} g(\gamma)$ , where  $0 < \underline{g} \leq \frac{1}{2}$ .

**Lemma A.13.** For any  $r > 0$ ,  $\Delta V(x, \sigma) > \bar{\delta}$  holds for all  $x \in [1 - 2\bar{\alpha}, 1]$  and  $\sigma \in \Sigma$ .

**Proof.** Since by feasibility,  $\phi(t, x, \sigma) \geq xe^{-t}$  for all  $t \geq 0$ ,  $x \in [0, 1]$ , and  $\sigma \in \Sigma$ , we have

$$\Delta V(x, \sigma) \geq (1+r) \int_0^\infty e^{-(1+r)t} \Delta u(xe^{-t}) dt$$

for all  $x \in [0, 1]$  and  $\sigma \in \Sigma$  by the monotonicity of  $\Delta u$ . The right hand side is increasing in  $x$  and  $r$ , and its value tends to  $\int_0^1 \Delta u(xy) dy$  as  $r \rightarrow 0$  (by the change of variables). Thus the desired conclusion follows by the above construction of  $\bar{\alpha}$ .  $\square$

This lemma immediately implies the following.

**Lemma A.14.** For any  $r > 0$  and any  $\eta \in (0, \bar{\alpha}]$ ,  $\Delta W(z, \sigma) > \bar{\delta}$  for all  $z \in [1 - \bar{\alpha}, 1 + \eta]$  and  $\sigma \in \Sigma$ .

Thus, if  $\eta \leq \bar{\alpha}$ , for any  $r > 0$ , any  $\bar{\delta}$ -equilibrium strategy plays action 1 for almost all signals  $z \in [1 - \bar{\alpha}, 1 + \eta]$ .

Let  $\sigma_{z^*}$  denote the  $z^*$ -strategy, the strategy that plays action 1 if and only if  $z \geq z^*$ . The mean-field dynamics generated by  $\sigma_{z^*}$  is given by  $\dot{x} = F^{\sigma_{z^*}}(x)$ , where

$$F^{\sigma_{z^*}}(x) = (E\sigma_{z^*})(x) - x = 1 - G\left(\frac{z^* - x}{\eta}\right) - x. \tag{A.4}$$

For each initial state  $x_0$ , let  $\phi_\eta(t, x_0, \sigma_{z^*})$  denote the path generated by (A.4).

For each  $x_0$ , define

$$T_{z^*, \eta}(x_0) = \inf \{T \mid (E\sigma_{z^*})(1 | \phi_\eta(t, x_0, \sigma_{z^*})) = 1 \text{ for all } t \geq T\},$$

where  $\inf \emptyset = \infty$  by convention. That is,  $T_{z^*, \eta}(x_0)$  is the infimum of time period  $T$  such that, under the path  $\phi_\eta(t, x_0, \sigma_{z^*})$ , all incoming agents choose action 1 after time  $T$ .

Define for each  $\tau \geq 0$ ,

$$\psi(t, x_0, \tau) = \begin{cases} x_0 e^{-t} & \text{if } t < \tau, \\ 1 - (1 - x_0 e^{-\tau}) e^{-(t-\tau)} & \text{if } t \geq \tau. \end{cases} \tag{A.5}$$

That is,  $\psi(\cdot, x_0, \tau)$  is the path from  $x_0$  that arises if all incoming agents choose action 0 before  $\tau$  and action 1 after  $\tau$ . Then by feasibility, for all  $x_0$ , we have  $\phi_\eta(t, x_0, \sigma_{z^*}) \geq \psi(t, x_0, \infty)$  for all  $t$  and hence  $\Delta V(x_0, \sigma_{z^*}) \geq (1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, x_0, \infty)) dt$ , while by construction, for  $x_0$  such that  $T_{z^*, \eta}(x_0) \leq \tau$ , we have  $\phi_\eta(t, x_0, \sigma_{z^*}) \geq \psi(t, x_0, \tau)$  for all  $t$  and hence  $\Delta V(x_0, \sigma_{z^*}) \geq (1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, x_0, \tau)) dt$ . Therefore, for any  $\tau \geq 0$ ,

$$\begin{aligned} \Delta W(z^*, \sigma_{z^*}) &\geq v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\}) \\ &\quad \times (1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, [z^* - \eta]_+, \tau)) dt \\ &\quad + (1 - v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\})) \\ &\quad \times (1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, [z^* - \eta]_+, \infty)) dt, \end{aligned} \tag{A.6}$$

where  $v_{z^*, \eta}$  denotes the agent's belief conditional on observing  $z^*$  under noise level  $\eta$ , and  $[z^* - \eta]_+ = \max\{z^* - \eta, 0\}$ .

The following lemma bounds the agent's belief under small  $\eta$ .

**Lemma A.15.** For any  $\tau > 0$ , there exists  $\bar{\eta} > 0$  with  $\bar{\eta} \leq \bar{\alpha}$  such that

$$v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\}) \geq 1 - z^* - \tau$$

for all  $\eta \in (0, \bar{\eta}]$  and  $z^* \in (-\eta, 1 - \bar{\alpha}]$ .

**Proof.** To simplify the exposition, we consider the case in which the prior belief is uniform. The general case is analogous; since the prior belief admits a continuous and positive density, the interim belief is approximately same as in the uniform prior case as  $\eta \rightarrow 0$ .

Fix any  $\tau > 0$ . Below we restrict attention to  $\eta$  strictly smaller than  $\underline{g}$  and  $\bar{\alpha}$ .

For each  $\eta \in (0, \bar{\alpha})$  and  $z^* \in (-\eta, 1 - \bar{\alpha}]$ , consider the equation

$$1 - G\left(\frac{z^* - x}{\eta}\right) = x. \tag{A.7}$$

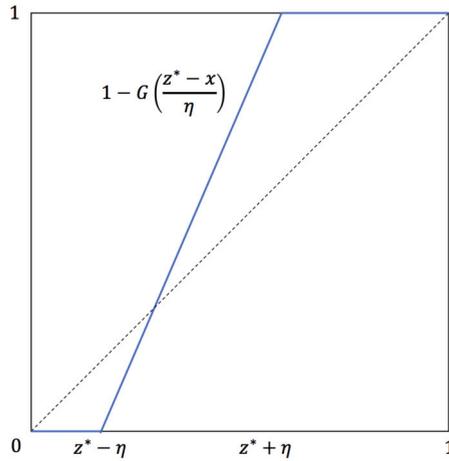


Fig. A.1.  $z^* \in (\eta, 1 - \bar{\alpha}]$ .

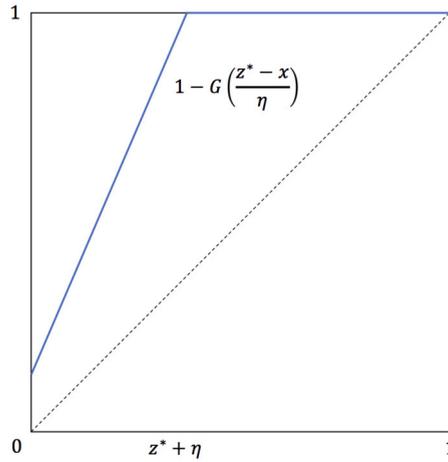


Fig. A.2.  $z^* \in (-\eta, \eta]$ .

The left hand side is equal to 0 if  $x \leq z^* - \eta$ , continuously increases at  $x \in [z^* - \eta, z^* + \eta]$  with its slope  $\frac{1}{\eta}g\left(\frac{z^*-x}{\eta}\right) \geq \frac{g}{\eta} > 1$ , and is equal to 1 if  $x \geq z^* + \eta$ . Therefore, this equation has at most one solution in  $(0, 1)$ . Specifically, if  $z^* \in (\eta, 1 - \bar{\alpha}]$  (resp.  $z^* \in (-\eta, \eta]$ ), then the equation has a unique solution (resp. no solution) in  $(0, 1)$ . See Figs. A.1 and A.2. Let  $x^*(\eta, z^*)$  be the solution in  $(0, 1)$  if  $z^* \in (\eta, 1 - \bar{\alpha}]$ , and let  $x^*(\eta, z^*) = 0$  if  $z^* \in (-\eta, \eta]$ . Note that in each case,  $x^*(\eta, z^*) \in [z^* - \eta, z^* + \eta]$ .

For each  $\eta \in (0, \bar{\alpha})$  and  $z^* \in (-\eta, 1 - \bar{\alpha}]$ , the distribution  $v_{z^*, \eta}$  is supported over  $[\max\{z^* - \eta, 0\}, z^* + \eta]$ , whose density at each  $x$  is  $\frac{g\left(\frac{z^*-x}{\eta}\right)}{\int_0^1 g\left(\frac{z^*-x'}{\eta}\right) dx'}$  since the prior belief over states is uniform. Thus  $v_{z^*, \eta}(\{x' \mid x' \geq x\}) = \frac{G\left(\frac{z^*-x}{\eta}\right)}{G\left(\frac{z^*}{\eta}\right)}$  for each  $x$ .

For each initial state  $x_0 > x^*(\eta, z^*)$ , the path  $\phi_\eta(t, x_0, \sigma_{z^*})$  is increasing in  $t$  (since  $F^{\sigma_{z^*}}(\phi_\eta(t, x_0, \sigma_{z^*})) > 0$  for all  $t$ ) and converges to 1 as  $t \rightarrow \infty$ . Thus,  $T_{z^*, \eta}(x_0)$  is the time at which the path reaches  $z^* + \eta$ . Since  $\frac{g}{\eta} > 1$ , the drift  $F^{\sigma_{z^*}}(\phi_\eta(t, x_0, \sigma_{z^*}))$  is strictly increasing over time until  $T_{z^*, \eta}(x_0)$ . Hence,

$$T_{z^*, \eta}(x_0) \leq \frac{z^* + \eta - x_0}{F^{\sigma_{z^*}}(\phi_\eta(0, x_0, \sigma_{z^*}))} = \frac{z^* + \eta - x_0}{1 - G\left(\frac{z^*-x_0}{\eta}\right) - x_0} < \frac{2\eta}{1 - G\left(\frac{z^*-x_0}{\eta}\right) - x_0} \tag{A.8}$$

for all  $z^* \in (-\eta, 1 - \bar{\alpha}]$  and all  $x_0 > x^*(\eta, z^*)$ , where the last inequality used  $x^* \in [z^* - \eta, z^* + \eta]$ .

We take  $\bar{\eta} > 0$  sufficiently small so that

$$\frac{2\bar{\eta}}{1 - G(0)} \leq \tau, \quad \bar{\eta} \leq \frac{\tau}{2}, \quad \frac{2\bar{\eta}}{(g - \bar{\eta})\epsilon} \leq \tau. \tag{A.9}$$

Below we fix  $\eta \in (0, \bar{\eta})$ , and bound the value of  $v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\})$ , depending on the location of  $z^*$ .

**Case 1:**  $z^* \in (-\eta, 0]$ : In this case

$$1 - G\left(\frac{z^* - x_0}{\eta}\right) - x_0 \geq 1 - G\left(\frac{z^*}{\eta}\right) \geq 1 - G(0)$$

for all  $x_0 \in [0, z^* + \eta]$ , where the first inequality follows as LHS is increasing in  $x_0$  in this range. Therefore by (A.8) and the first inequality in (A.9), we have  $v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\}) = 1$  since  $v_{z^*, \eta}$  is supported on  $[0, z^* + \eta]$ .

**Case 2:**  $z^* \in [0, 1 - \bar{\alpha}]$ : Take an  $\epsilon > 0$  such that  $\epsilon \bar{g} < \min\{G(0)\tau, \frac{\tau}{2}\}$ . Observe that  $1 - G\left(\frac{z^* - x}{\eta}\right) - x$  is greater than or equal to 0 at  $x = x^*(\eta, z^*)$  and is increasing in  $x$  with the slope  $\frac{1}{\eta}g\left(\frac{z^* - x}{\eta}\right) - 1 \geq \frac{g}{\eta} - 1$  at  $x \in [z^* - \eta, z^* + \eta]$ . Therefore, for all  $z^* \in (\eta, 1 - \bar{\alpha}]$ , if  $x_0 \in [x^*(\eta, z^*) + \eta\epsilon, z^* + \eta]$ , then we have

$$1 - G\left(\frac{z^* - x_0}{\eta}\right) - x_0 \geq \left(\frac{g}{\eta} - 1\right)\eta\epsilon = (g - \eta)\epsilon.$$

Thus, by (A.8) and the third inequality in (A.9), we have  $T_{z^*, \eta}(x_0) \leq \tau$  whenever  $x_0 \in [x^*(\eta, z^*) + \eta\epsilon, z^* + \eta]$ . Suppose first that  $z^* \in [0, \eta]$ , where  $x^*(\eta, z^*) = 0$ . Then we have

$$\begin{aligned} v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\}) &\geq v_{z^*, \eta}([\eta\epsilon, z^* + \eta]) = \frac{G\left(\frac{z^* - \eta\epsilon}{\eta}\right)}{G\left(\frac{z^*}{\eta}\right)} \\ &\geq \frac{G\left(\frac{z^*}{\eta}\right) - \epsilon \bar{g}}{G\left(\frac{z^*}{\eta}\right)} \\ &\geq 1 - \frac{\epsilon \bar{g}}{G(0)} \\ &\geq 1 - \tau \geq 1 - z^* - \tau, \end{aligned}$$

where the third inequality follows from the choice of  $\epsilon$ . Suppose next that  $z^* \in (\eta, 1 - \bar{\alpha}]$ , where  $1 - G\left(\frac{z^* - x^*(\eta, z^*)}{\eta}\right) = x^*(\eta, z^*)$  by (A.7). Then we have

$$\begin{aligned} v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\}) &\geq v_{z^*, \eta}([x^*(\eta, z^*) + \eta\epsilon, z^* + \eta]) = G\left(\frac{z^* - x^*(\eta, z^*) - \eta\epsilon}{\eta}\right) \\ &\geq G\left(\frac{z^* - x^*(\eta, z^*)}{\eta}\right) - \frac{\tau}{2} \\ &= 1 - x^*(\eta, z^*) - \frac{\tau}{2} \\ &\geq 1 - z^* - \tau, \end{aligned}$$

where the second inequality follows from the choice of  $\epsilon$ , and the last inequality holds since  $z^* - x^*(\eta, z^*) \geq -\eta \geq -\frac{\tau}{2}$ .  $\square$

**Proof of Proposition 4.** Define the continuous function

$$\begin{aligned} w(p, x, \tau, r) &= p(1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, x, \tau)) dt \\ &\quad + (1-p)(1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, x, \infty)) dt, \end{aligned}$$

where  $\psi$  is as defined in (A.5). For each  $x \in [0, 1]$ , observe that

$$w(1-x, x, 0, r) \rightarrow \int_0^1 \Delta u(y) dy = 2\bar{\delta}$$

as  $r \rightarrow 0$ . The convergence is uniform in  $x$ .<sup>22</sup> We also observe that  $w(p, x, \tau, r)$  is Lipschitz continuous in  $(x, \tau)$  (uniformly across  $p, r$ ). Thus, there exist  $\bar{r} > 0$  and  $\tau > 0$  with  $\tau \leq \bar{\alpha}$  such that

<sup>22</sup> To see this, observe that  $(1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, x, 0)) dt$  and  $(1+r) \int_0^\infty e^{-(1+r)t} \Delta u(\psi(t, x, \infty)) dt$  are decreasing/increasing in  $r$  respectively by the monotonicity of  $\Delta u$ , so their convergences as  $r \rightarrow 0$  are uniform across  $x \in [0, 1]$ .

$$w(1 - z - \tau, [z - \tau]_+, \tau, r) > \bar{\delta}$$

for all  $r \in (0, \bar{r}]$  and all  $z \in [-\tau, 1 - \tau]$ . Given such a  $\tau$ , let  $\bar{\eta} \in (0, \bar{\alpha}]$  be as in Lemma A.15, where we assume without loss that  $\bar{\eta} \leq \tau$ . Then, if  $\eta \in (0, \bar{\eta}]$  and  $r \in (0, \bar{r}]$ , then we have

$$\begin{aligned} \Delta W(z^*, \sigma_{z^*}) &\geq w(v_{z^*, \eta}(\{x \mid T_{z^*, \eta}(x) \leq \tau\}), [z - \tau]_+, \tau, r) \\ &\geq w(1 - z - \tau, [z - \tau]_+, \tau, r) > \bar{\delta} \end{aligned} \tag{A.10}$$

for all  $z^* \in (-\eta, 1 - \bar{\alpha}]$  by (A.6) and Lemma A.15. By Lemma A.14, (A.10) in fact holds for all  $z \in (-\eta, 1 + \eta]$ .

Now let  $\eta \in (0, \bar{\eta}]$  and  $r \in (0, \bar{r}]$ . For the “always-1” strategy, or  $\sigma_{-\eta}$ , we have

$$\Delta W(z, \sigma_{-\eta}) \geq w(1, 0, 0, r) > \bar{\delta}$$

for all  $z$  by the choice of  $\bar{r}$ , which implies that it is a  $\bar{\delta}$ -equilibrium. Then for uniqueness, suppose to the contrary that there exists a  $\bar{\delta}$ -equilibrium  $\sigma$  such that  $\sigma(0|z) > 0$  holds for a set of  $z$ 's with a positive measure, so that  $\{z \mid \Delta W(z, \sigma) \leq \bar{\delta}\}$  has a positive measure. Let  $z^* = \sup\{z \mid \Delta W(z, \sigma) \leq \bar{\delta}\}$ . Observe that  $\Delta W(\cdot, \sigma)$  is continuous by the continuity of  $\Delta V(\cdot, \sigma)$  and  $g$ . Thus, we have  $\Delta W(z^*, \sigma) \leq \bar{\delta}$ . However, since  $\sigma(0|z) = 0$  for almost all  $z > z^*$  and hence  $\phi(t, x, \sigma_{z^*}) \leq \phi(t, x, \sigma)$  for all  $t$  and  $x$ , we have  $\Delta W(z^*, \sigma_{z^*}) \leq \Delta W(z^*, \sigma) \leq \bar{\delta}$ , which contradicts (A.10).  $\square$

## Appendix B. Details of the examples

### B.1. Details of Examples 1 and 2

To simplify notation below we identify  $x \in \Delta$  with  $x_1 \in [0, 1]$ , i.e., the fraction of action 1.

For  $\kappa \in [0, 1]$ , suppose that the agents follow the  $\kappa$ -strategy, the strategy that plays action 1 if and only if  $x_1 \geq \kappa$ . Let  $X_1(t)$  denote the number of agents who are playing action 1 at time  $t$ , where  $X_1(0) = k$  is given. If  $\frac{k}{N} < \kappa$  ( $\frac{k}{N} \geq \kappa$ , resp.), then every entrant agent plays action 0 (1, resp.), so that  $\mathbb{E}[X_1(t) \mid X_1(0) = k]$  ( $\mathbb{E}[N - X_1(t) \mid X_1(0) = k]$ , resp.) is equal to  $k$  ( $N - k$ , resp.) times the probability that the Poisson clock does not ring before  $t$ , which is  $e^{-t}$ . We thus have the following formula:

$$(1+r) \int_0^\infty e^{-(1+r)t} \mathbb{E} \left[ x_1(t) \mid x_1(0) = \frac{k}{N} \right] dt = \begin{cases} \frac{1+r}{2+r} \frac{k}{N} & \text{if } \frac{k}{N} < \kappa \\ 1 - \frac{1+r}{2+r} \frac{N-k}{N} & \text{if } \frac{k}{N} \geq \kappa, \end{cases} \tag{B.1}$$

where  $x_1(t) = \frac{X_1(t)}{N}$ .

#### B.1.1. Example 1

We consider the 1-strategy ( $\kappa$ -strategy with  $\kappa = 1$ ), in which any agent chooses 1 if all the agents in the population are using 1, and 0 otherwise.

**Proposition B.2.** *The 1-strategy is an equilibrium if and only if  $\frac{1}{N} \leq c \leq \frac{1}{2+r} + \frac{1+r}{2+r} \frac{1}{N}$ .*

Thus, the 1-strategy is an equilibrium in particular when  $c < \frac{1}{2}$ ,  $r \leq \frac{1}{c} - 2$ , and  $N \geq \frac{1}{c}$ .

**Proof.** Given the population size  $N$ , let  $V_N(a, x'_1)$  denote the new entrant's expected life time payoff to action  $a$  when the other agents follow the 1-strategy and the state after the choice of the entrant is  $x'_1$ . We readily have

$$V_N(1, 1) = 1 - c$$

and, by (B.1),

$$V_N\left(0, \frac{k}{N}\right) = \frac{1+r}{2+r} \frac{k}{N}, \quad V_N\left(1, \frac{k}{N}\right) = V_N\left(0, \frac{k}{N}\right) - c$$

for each  $k \leq N - 1$ .

To verify that the above strategy is an equilibrium, first we consider the new entering agent's incentive upon observing  $x_1 = 1$ . Choosing action 1 is optimal if and only

$$V_N(1, 1) \geq V_N\left(0, \frac{N-1}{N}\right),$$

or  $c \leq \frac{1}{2+r} + \frac{1+r}{2+r} \frac{1}{N}$ .

Next we consider the incentive of the new entrant who observes  $x_1 = \frac{N-1}{N}$ . In this case, choosing action 0 is optimal if and only if

$$\frac{1}{N} V_N\left(0, \frac{N-1}{N}\right) + \frac{N-1}{N} V_N\left(0, \frac{N-2}{N}\right) \geq \frac{1}{N} V_N(1, 1) + \frac{N-1}{N} V_N\left(1, \frac{N-1}{N}\right),$$

or  $c \geq \frac{1}{N}$ .

Finally, if  $x_1 = \frac{k}{N}$  with  $k \leq N - 2$ , then choosing action 0 is optimal if and only

$$\frac{N-k}{N} V_N \left( 0, \frac{k}{N} \right) + \frac{k}{N} V_N \left( 0, \frac{k-1}{N} \right) \geq \frac{N-k}{N} V_N \left( 1, \frac{k+1}{N} \right) + \frac{k}{N} V_N \left( 1, \frac{k}{N} \right),$$

or  $c \geq \frac{1+r}{2+r} \frac{1}{N}$ , which holds whenever  $c \geq \frac{1}{N}$ .

Hence, the 1-strategy is an equilibrium if and only if  $\frac{1}{N} \leq c \leq \frac{1}{2+r} + \frac{1+r}{2+r} \frac{1}{N}$ .  $\square$

**B.1.2. Example 2**

**Proposition B.3.**

- (1) If  $0 < c \leq \frac{1}{2+r}$ , then “always play 1” is the unique equilibrium for all  $N$ .
- (2) If  $\frac{1}{2+r} < c < \frac{1+r}{2+r}$ , then for any  $\bar{\kappa} \in [\frac{(2+r)c-1}{1+r}, \frac{(2+r)c-1}{r}]$ , there exists  $N_0$  such that for any  $\kappa \in [\frac{(2+r)c-1}{1+r}, \bar{\kappa}]$ , the  $\kappa$ -strategy is an equilibrium for all  $N \geq N_0$ .
- (3) If  $\frac{1+r}{2+r} \leq c < 1$ , then there exists  $N_0$  such that for any  $\kappa \in [\frac{(2+r)c-1}{1+r}, 1]$ , the  $\kappa$ -strategy is an equilibrium for all  $N \geq N_0$ , and “always play 0” is an equilibrium for all  $N$ .

**Proof.** For a given  $\kappa$ , let  $V_N(a, x'_1)$  denote a new entrant’s expected life time payoff to action  $a$  when the other agents follow the  $\kappa$ -strategy and the state after the choice of the entrant is  $x'_1$ . We have

$$V_N(0, x'_1) = c$$

and, by (B.1),

$$V_N(1, x'_1) = \begin{cases} \frac{1+r}{2+r} x'_1 & \text{if } x'_1 < \kappa \\ 1 - \frac{1+r}{2+r} (1 - x'_1) & \text{if } x'_1 \geq \kappa \end{cases}$$

for each  $x'_1 \in \{0, \frac{1}{N}, \dots, 1\}$ .

First, if  $x_1 = 1$  is observed, then clearly it is optimal to choose action 1 by  $c < 1$ . Then, consider the incentive of a new entrant who observes  $x_1 \in [\kappa, \frac{N-1}{N}]$ . It is optimal to choose 1 if and only if

$$x_1 V_N(1, x_1) + (1 - x_1) V_N(1, x_1 + \frac{1}{N}) \geq c,$$

or

$$1 - \frac{1+r}{2+r} \frac{N-1}{N} (1 - x_1) \geq c. \tag{B.2}$$

This condition holds for all  $N$  if and only if  $1 - \frac{1+r}{2+r} (1 - x_1) \geq c$ , or  $x_1 \geq \frac{(2+r)c-1}{1+r}$ .

Next, consider the case where a new entrant observes  $x_1 \in [\kappa - \frac{1}{N}, \kappa)$ . It is optimal to choose 0 if and only if

$$x_1 V_N(1, x_1) + (1 - x_1) V_N(1, x_1 + \frac{1}{N}) \leq c,$$

or

$$\frac{1+r}{2+r} - \left( \frac{r}{2+r} - \frac{1+r}{2+r} \frac{1}{N} \right) (1 - x_1) \leq c. \tag{B.3}$$

Last, consider the case where a new entrant observes  $x_1 < \kappa - \frac{1}{N}$ . It is optimal to choose 0 if and only if

$$x_1 V_N(1, x_1) + (1 - x_1) V_N(1, x_1 + \frac{1}{N}) \leq c,$$

or

$$\frac{1+r}{2+r} - \frac{1+r}{2+r} \frac{N-1}{N} (1 - x_1) \leq c. \tag{B.4}$$

(1) Suppose that  $c \leq \frac{1}{2+r}$ . In this case, (B.2) holds for all  $N$  and all  $x_1 \in [0, 1]$ . This implies that the 0-strategy ( $\kappa$ -strategy with  $\kappa = 0$ ) is an equilibrium for all  $N$ .

Consider any  $\kappa$ -strategy with  $\kappa > 0$ ; set  $\kappa \geq \frac{1}{N}$  without loss. The left hand side of (B.3) is greater than or equal to  $\min\{\frac{1}{2+r} + \frac{1+r}{2+r} \frac{1}{N}, \frac{1+r}{2+r}\}$ , which is greater than  $c$  when  $c \leq \frac{1}{2+r}$ . This implies that this strategy is not an equilibrium. Thus, by a usual monotonicity argument, there is no equilibrium that plays action 0 with positive probability.

(2) Suppose that  $\frac{1}{2+r} < c < \frac{1+r}{2+r}$ . Let  $\bar{\kappa} < \frac{(2+r)c-1}{r}$ . In this case, since  $\frac{1+r}{2+r} - \frac{r}{2+r} (1 - \bar{\kappa}) < c$ , there exists  $N_1$  such that (B.3) holds for all  $N \geq N_1$  and all  $x_1 \leq \bar{\kappa}$ , while since  $\frac{1+r}{2+r} - \frac{1+r}{2+r} (1 - \bar{\kappa}) = \frac{1+r}{2+r} \bar{\kappa} < \frac{1+r}{2+r} \frac{(2+r)c-1}{r} = \frac{1+r}{r} (c - \frac{1}{2+r}) < c$ , there exists  $N_2$  such that (B.4) holds

for all  $N \geq N_2$  and all  $x_1 \leq \bar{\kappa}$ . Let  $N_0 = \max\{N_1, N_2\}$ . Take any  $\kappa \in [\frac{(2+r)c-1}{1+r}, \bar{\kappa}]$ , and consider the  $\kappa$ -strategy. Then (B.2) holds for all  $N$ , while (B.3) and (B.4) hold for all  $N \geq N_0$ . This implies that this strategy is an equilibrium if  $N \geq N_0$ .

(3) Suppose that  $c \geq \frac{1+r}{2+r}$ . Take any  $\kappa \in [\frac{(2+r)c-1}{1+r}, 1]$ , and consider the  $\kappa$ -strategy. In this case, (B.2) and (B.4) hold for all  $N$ , while (B.3) holds whenever  $N \geq \frac{1+r}{r}$ . This implies that this strategy is an equilibrium whenever  $N \geq \frac{1+r}{r}$ .

That (B.4) holds for all  $N$  also implies that in this case, “always play 0” is an equilibrium for all  $N$ .  $\square$

Blume (2005) considers a related coordination game model in which each agent is infinitely lived and can adjust her action upon a Poisson shock. In contrast to our equilibrium multiplicity result, Blume (2005) shows that his model admits a unique Markov equilibrium as agents become sufficiently patient. Our model differs from his in two respects: (i) each agent commits to an action throughout her lifetime, and (ii) each agent leaves the population upon a Poisson shock (with normalized rate 1). Indeed, (i) is the main source of the difference in the results of the two models, while (ii) is inessential. Infinite lifetime can effectively be incorporated in our model by sending the discount rate  $r$  to  $-1$  (since our model is well defined as long as the “effective” discount rate  $1+r$  is positive), where multiplicity will still remain: by the proof of Proposition B.3 above, one can verify that for any  $\kappa \in (0, 1-c)$ , the  $\kappa$ -strategy is an equilibrium for all  $r > -1$  sufficiently close to  $-1$  and all  $N$ .

B.2. Details of Examples 4 and 5

B.2.1. Assumption L

We show that Assumption L is satisfied in the setting of Example 4. Let  $\Gamma$  denote the support of  $g$ , which is assumed to be convex and compact. Take any measurable set  $B \subseteq \mathbb{R}^{|A|-1}$ . Observe

$$\mu_x(B) = \int_{\Gamma} 1_B(x + \eta\gamma)g(\gamma)d\gamma = \int_{x+\eta\Gamma} 1_B(z)g\left(\frac{z-x}{\eta}\right)\frac{1}{\eta}dz.$$

Thus for any  $x, x' \in \Delta$ ,

$$\begin{aligned} |\mu_x(B) - \mu_{x'}(B)| &\leq \frac{1}{\eta} \int_{(x+\eta\Gamma) \cap (x'+\eta\Gamma)} \left| g\left(\frac{z-x}{\eta}\right) - g\left(\frac{z-x'}{\eta}\right) \right| dz \\ &\quad + \frac{1}{\eta} \lambda((x+\eta\Gamma) \cup (x'+\eta\Gamma) \setminus ((x+\eta\Gamma) \cap (x'+\eta\Gamma))) \sup_{\gamma \in \Gamma} g(\gamma), \end{aligned}$$

where  $\lambda$  is the Lebesgue measure on  $\mathbb{R}^{|A|-1}$ . The first term is bounded by  $\frac{1}{\eta} \lambda(\Gamma) \|x - x'\|$  times the Lipschitz coefficient of  $g$ . To see how the second term is bounded by  $\|x - x'\|$  times a constant, note that

$$\begin{aligned} \lambda((x+\eta\Gamma) \cup (x'+\eta\Gamma)) &\leq \lambda\left(\left\{y \mid \inf_{y' \in x+\eta\Gamma} \|y - y'\|_2 \leq \|x - x'\|_2\right\}\right) \\ &\leq \lambda(x+\eta\Gamma) + \|x - x'\|_2 K \end{aligned}$$

where  $K \geq 0$  is a constant that depends only on  $\eta\Gamma$ . The existence of such  $K$  follows from the fact that the right hand side in the first line is written as a polynomial of order  $(|A| - 1)$  in  $\|x - x'\|_2$  by the Steiner formula and that  $\Delta$  is bounded. Likewise  $\lambda((x+\eta\Gamma) \cup (x'+\eta\Gamma)) \leq \lambda(x'+\eta\Gamma) + \|x - x'\|_2 K$ . Thus

$$\begin{aligned} \lambda((x+\eta\Gamma) \cup (x'+\eta\Gamma)) - \lambda((x+\eta\Gamma) \cap (x'+\eta\Gamma)) \\ = 2\lambda((x+\eta\Gamma) \cup (x'+\eta\Gamma)) - \lambda(x+\eta\Gamma) - \lambda(x'+\eta\Gamma) \leq 2\|x - x'\|_2 K, \end{aligned}$$

which is bounded by  $2\|x - x'\|K$ .

B.2.2. Assumption U

We show that Assumption U is satisfied in the setting of Example 5. To simplify the exposition, we consider only the case where  $A = \{0, 1\}$ ; the general case can be proved analogously. As in Section 4, we let  $x \in [0, 1]$  denote the fraction of action-1 agents. The signal is of the form  $z = x + \eta\gamma$ , where  $\gamma$  follows a (Lipschitz) continuous density  $g$  with compact convex support  $\Gamma$ . For simplicity, we assume that  $\eta > 0$  is small enough that  $\eta < \frac{1}{2}$  and that  $\Gamma = [-1, 1]$ ; thus,  $Z = [-\eta, 1 + \eta]$ . We restrict attention to  $N \geq \frac{1}{2\eta}$ , so that  $\bigcup_{x \in [0, 1]} [x - \eta, x + \eta] = Z$ . Assume that  $\nu_N$  converges weakly as  $N \rightarrow \infty$  to the prior  $\nu$  and that  $\nu$  admits a density  $p$  strictly positive and continuous on  $[0, 1]$ . By Rao (1962, Theorem 4.2), it follows that  $\nu_N(I) \rightarrow \nu(I)$  as  $N \rightarrow \infty$  uniformly over all intervals  $I \subset [0, 1]$ .

Fix any continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ , and for  $z \in Z$ , write

$$h_N(z) = \int_{x \in [0, 1]} f(x) d\nu_{N,z}(x)$$

$$= \frac{\int_{x \in [0,1] \cap [z-\eta, z+\eta]} f(x)g\left(\frac{z-x}{\eta}\right) d\nu_N(x)}{\int_{x \in [0,1] \cap [z-\eta, z+\eta]} g\left(\frac{z-x}{\eta}\right) d\nu_N(x)}$$

(where  $\nu_N$  is viewed as a distribution on  $[0, 1]$  with  $\text{supp } \nu_N \subset [0, 1]_N$ ), and

$$h(z) = \int_{x \in [0,1]} f(x)d\nu_z(x) = \begin{cases} \frac{\int_{x \in [0,1] \cap [z-\eta, z+\eta]} f(x)g\left(\frac{z-x}{\eta}\right) d\nu(x)}{\int_{x \in [0,1] \cap [z-\eta, z+\eta]} g\left(\frac{z-x}{\eta}\right) d\nu(x)} & \text{if } z \in (-\eta, 1 + \eta), \\ f(0) & \text{if } z = -\eta, \\ f(1) & \text{if } z = 1 + \eta. \end{cases}$$

For  $\epsilon > 0$ , let  $\delta(\epsilon) > 0$  be such that if  $|x - x'| \leq \delta(\epsilon)$ ,  $x, x' \in [0, 1]$  and  $|\gamma - \gamma'| \leq \frac{\delta(\epsilon)}{\eta}$ ,  $\gamma, \gamma' \in [-1, 1]$ , then  $|g(\gamma) - g(\gamma')| \leq \epsilon$  and  $|f(x)g(\gamma) - f(x')g(\gamma')| \leq \epsilon$ . Let  $\{I_k\}_{k=1}^{K(\epsilon)}$  be a partition of  $[0, 1]$  such that each  $I_k$  is a (nontrivial) interval of length less than  $\delta(\epsilon)$ . For each  $k$ , take any  $x_k \in I_k$ .

**Claim 1.** For any  $\epsilon > 0$  and  $N \geq \frac{1}{2\eta}$ ,  $|h_N(z) - h(z)| \leq 2\epsilon$  whenever  $-\eta \leq z < -\eta + \delta(\epsilon)$  or  $1 + \eta - \delta(\epsilon) < z \leq 1 + \eta$ .

**Proof.** Assume the former case  $-\eta \leq z < -\eta + \delta(\epsilon)$  (an analogous argument applies to the latter case). In this case, if  $x \in [0, 1] \cap [z - \eta, z + \eta]$ , then  $|x - 0| \leq \delta(\epsilon)$ , so that  $|f(x) - f(0)| \leq \epsilon$  by the choice of  $\delta(\epsilon)$ . For any  $N \geq \frac{1}{2\eta}$ , we have  $|h_N(z) - h(z)| \leq |h_N(z) - f(0)| + |f(0) - h(z)| \leq 2\epsilon$ .  $\square$

For  $z \in (-\eta, 1 + \eta)$ , write

$$D_N(z) = \int_{x \in [0,1] \cap [z-\eta, z+\eta]} f(x)g\left(\frac{z-x}{\eta}\right) d\nu_N(x),$$

$$E_N(z) = \int_{x \in [0,1] \cap [z-\eta, z+\eta]} g\left(\frac{z-x}{\eta}\right) d\nu_N(x),$$

$$D(z) = \int_{x \in [0,1] \cap [z-\eta, z+\eta]} f(x)g\left(\frac{z-x}{\eta}\right) d\nu(x),$$

$$E(z) = \int_{x \in [0,1] \cap [z-\eta, z+\eta]} g\left(\frac{z-x}{\eta}\right) d\nu(x).$$

For  $\epsilon > 0$ , let  $N_1(\epsilon)$  be any natural number such that if  $N \geq N_1(\epsilon)$ , then

$$|\nu_N(I) - \nu(I)| \leq \frac{\epsilon}{K(\epsilon)}$$

for all intervals  $I \subset [0, 1]$ . Denote  $f_{\max} = \max_{x \in [0,1]} |f(x)|$  and  $g_{\max} = \max_{\gamma \in [-1,1]} g(\gamma)$ .

**Claim 2.** For any  $\epsilon > 0$ , if  $N \geq N_1(\epsilon)$ , then  $|D_N(z) - D(z)| \leq (2 + f_{\max}g_{\max})\epsilon$  and  $|E_N(z) - E(z)| \leq (2 + g_{\max})\epsilon$  whenever  $-\eta < z < 1 + \eta$ .

**Proof.** Assume that  $N \geq N_1(\epsilon)$  and  $-\eta < z < 1 + \eta$ . Then we have

$$|D_N(z) - D(z)| \leq \sum_{k=1}^{K(\epsilon)} \int_{x \in I_k \cap [z-\eta, z+\eta]} \left| f(x)g\left(\frac{z-x}{\eta}\right) - f(x_k)g\left(\frac{z-x_k}{\eta}\right) \right| d\nu_N(x) + \sum_{k=1}^{K(\epsilon)} \int_{x \in I_k \cap [z-\eta, z+\eta]} \left| f(x)g\left(\frac{z-x}{\eta}\right) - f(x_k)g\left(\frac{z-x_k}{\eta}\right) \right| d\nu(x) + \sum_{k=1}^{K(\epsilon)} |f(x_k)g\left(\frac{z-x_k}{\eta}\right)| |\nu_N(I_k \cap [z-\eta, z+\eta]) - \nu(I_k \cap [z-\eta, z+\eta])|$$

$$\leq (2 + f_{\max} g_{\max})\epsilon,$$

where the second inequality follows from the choice of  $\delta(\epsilon)$  and  $N_1(\epsilon)$ , since if  $x \in I_k$ , then  $|x - x_k| \leq \delta(\epsilon)$  and  $\left| \frac{z-x}{\eta} - \frac{z-x_k}{\eta} \right| \leq \frac{|x-x_k|}{\eta} \leq \frac{\delta(\epsilon)}{\eta}$ . Letting  $f \equiv 1$ , we also have  $|E_N(z) - E(z)| \leq (2 + g_{\max})\epsilon$ .  $\square$

Next we want to obtain a positive uniform lower bound for  $E(z)$ . Note that

$$\begin{aligned} E(z) &= \int_{x \in [0,1] \cap [z-\eta, z+\eta]} g\left(\frac{z-x}{\eta}\right) p(x) dx \\ &\geq p_{\min} \int_{x \in [0,1] \cap [z-\eta, z+\eta]} g\left(\frac{z-x}{\eta}\right) dx, \end{aligned}$$

where  $p_{\min} = \min_{x \in [0,1]} p(x) > 0$  by assumption. For  $\delta > 0$ , denote

$$\underline{E}(\delta) = \eta p_{\min} \min \left\{ \int_{\gamma \in [-1, -1 + \frac{\delta}{\eta}]} g(\gamma) d\gamma, \int_{\gamma \in [1 - \frac{\delta}{\eta}, 1]} g(\gamma) d\gamma \right\} \in (0, \eta p_{\min}].$$

**Claim 3.** For any  $\delta > 0$ ,  $E(z) \geq \underline{E}(\delta)$  whenever  $-\eta + \delta \leq z \leq 1 + \eta - \delta$ .

**Proof.** Assume that  $-\eta + \delta \leq z \leq 1 + \eta - \delta$ . If  $[z - \eta, z + \eta] \subset [0, 1]$ , then  $E(z) \geq \eta p_{\min} \geq \underline{E}(\delta)$ . If  $z - \eta < 0$  (where  $z + \eta < 1$  by  $\eta < \frac{1}{2}$ ), then  $\frac{z-x}{\eta} \in [-1, -1 + \frac{\delta}{\eta}]$  implies  $x \in [0, 1] \cap [z - \eta, z + \eta]$ , so that  $E(z) \geq \eta p_{\min} \int_{\gamma \in [-1, -1 + \frac{\delta}{\eta}]} g(\gamma) d\gamma$ . If  $z + \eta > 1$  (where  $z - \eta > 0$  by  $\eta < \frac{1}{2}$ ), then  $\frac{z-x}{\eta} \in [1 - \frac{\delta}{\eta}, 1]$  implies  $x \in [0, 1] \cap [z - \eta, z + \eta]$ , so that  $E(z) \geq \eta p_{\min} \int_{\gamma \in [1 - \frac{\delta}{\eta}, 1]} g(\gamma) d\gamma$ . Thus, in each case, we have  $E(z) \geq \underline{E}(\delta)$ .  $\square$

By Claims 2 and 3, we have:

**Claim 4.** For any  $\delta > 0$ , if  $N \geq N_1\left(\frac{\underline{E}(\delta)}{2(2 + g_{\max})}\right)$ , then  $E_N(z) \geq \frac{\underline{E}(\delta)}{2}$  whenever  $-\eta + \delta \leq z \leq 1 + \eta - \delta$ .

For  $\epsilon > 0$  and  $\delta > 0$ , let

$$N_2(\epsilon, \delta) = \max \left\{ N_1(\epsilon), N_1\left(\frac{\underline{E}(\delta)}{2(2 + g_{\max})}\right) \right\}$$

and

$$C(\delta) = \frac{2(2 + f_{\max} g_{\max})}{\underline{E}(\delta)} + \frac{2f_{\max} g_{\max}(2 + g_{\max})}{\underline{E}(\delta)^2}.$$

By Claims 2-4, we have:

**Claim 5.** For any  $\epsilon > 0$  and  $\delta > 0$ , if  $N \geq N_2(\epsilon, \delta)$ , then  $|h_N(z) - h(z)| \leq C(\delta)\epsilon$  whenever  $-\eta + \delta \leq z \leq 1 + \eta - \delta$ .

**Proof.** Assume that  $N \geq N_2(\epsilon, \delta)$  and  $-\eta + \delta \leq z \leq 1 + \eta - \delta$ . Then by Claims 2-4, we have

$$\begin{aligned} |h_N(z) - h(z)| &= \left| \frac{D_N(z)}{E_N(z)} - \frac{D(z)}{E(z)} \right| \leq \frac{|D_N(z) - D(z)|}{E_N(z)} + \frac{|D(z)||E(z) - E_N(z)|}{E_N(z)E(z)} \\ &\leq \left[ \frac{2(2 + f_{\max} g_{\max})}{\underline{E}(\delta)} + \frac{2f_{\max} g_{\max}(2 + g_{\max})}{\underline{E}(\delta)^2} \right] \epsilon, \end{aligned}$$

as claimed.  $\square$

Finally, let

$$N_3(\epsilon) = N_2\left(\frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}}, \delta\left(\frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}}\right)\right).$$

**Claim 6.** For any  $\epsilon > 0$ , if  $N \geq N_3(\epsilon)$ , then  $|h_N(z) - h(z)| \leq \epsilon$  for all  $z \in Z$ .

**Proof.** Assume that  $N \geq N_3(\epsilon)$ . If  $-\eta \leq z < -\eta + \delta \left( \frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}} \right)$  or  $1 + \eta - \delta \left( \frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}} \right) < z \leq 1 + \eta$ , then  $|h_N(z) - h(z)| \leq 2 \frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}} \leq \epsilon$  by Claim 1. If  $-\eta + \delta \left( \frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}} \right) \leq z \leq 1 + \eta - \delta \left( \frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}} \right)$ , then  $|h_N(z) - h(z)| \leq C(\delta(\epsilon)) \frac{\epsilon}{\max\{2, C(\delta(\epsilon))\}} \leq \epsilon$  by Claim 5.  $\square$

## Data availability

No data was used for the research described in the article.

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