STRICT ROBUSTNESS TO INCOMPLETE INFORMATION: ADDENDUM

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Abstract. In Morris et al. (2023a), we proposed and studied the notion of strict robustness, a strict version of robustness of Kajii and Morris (1997). This note discusses another strict version that had been introduced earlier by Oury and Tercieux (2007). By definition it is weaker than ours, and the two versions are equivalent in binary-action supermodular games. In many-action supermodular games, they are equivalent if the perturbations of the respective versions are restricted to be supermodular.

1. Introduction

In a broad sense, an equilibrium of a complete information game is robust if every "nearby" incomplete information game has an equilibrium that induces an action distribution close to the original equilibrium (Kajii and Morris, 1997). In the original definition of Kajii and Morris (1997), "nearby" incomplete information games are games in which with high probability, players know that their payoff functions are precisely equal to those of the complete information game. In Morris et al. (2023a), we proposed a strict version of robustness, *strict robustness*, by allowing for a larger class of "nearby" incomplete information games, where with high probability, players know that their payoffs are close to those of the complete information game *in expectation*. We showed that unique correlated equilibrium and strict monotone potential maximizer (if the game or the strict monotone potential is supermodular) are each sufficient for strict robustness, and the latter is also necessary in binary-action supermodular (BAS) games.

In this note, we discuss a similar strict robustness notion that had been introduced earlier by Oury and Tercieux (2007), which we failed to acknowledge in Morris et al.

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Date: April 21, 2025.
Keywords. Incomplete information, robustness, supermodular game, information design.
Stephen Morris gratefully acknowledges financial support from NSF Grant SES-2049744 and SES-1824137. Daisuke Oyama gratefully acknowledges financial support from JSPS KAKENHI Grants 18KK0359 and 23K01303.

(2023a). In their version, "nearby" incomplete information games are defined as games in which with high probability, players know that their payoffs are close to those of the complete information game *with probability one*. By definition, "nearby" games in their sense are "nearby" games in our sense, and therefore, their strict robustness notion is weaker than ours; thus we refer to their notion as *weak strict robustness* here—see Section 2 for the formal definitions. Clearly, our sufficiency results mentioned above apply also to weak strict robustness. The results of Oury and Tercieux (2007) are summarized in Section 3.

In Morris et al. (2023a), we adopted our definitions of "nearby" incomplete information games and strict robustness in order to establish a clean connection with our study on implementation by information design (Morris et al., 2024). Indeed, in proving our necessity result for BAS games, we relied on the full implementation result of Morris et al. (2024), where the information structures constructed to implement a target outcome are "nearby" games in our sense as the prior distribution approaches the complete information limit. However, it is also possible to construct "nearby" games in the sense of Oury and Tercieux (2007) to implement such an outcome, as we will formally describe in Section 4. Together with existing results, this implies that strict robustness and weak strict robustness are equivalent in BAS games.

Section 5 presents our new results on the robustness notions when "nearby" games of the two versions are restricted to be supermodular—strict robustness and weak strict robustness to supermodular elaborations. For many-action supermodular games, by extending the construction given in Oury and Tercieux (2007) we establish the existence of and characterize the smallest set that is (weakly) strictly robust to supermodular elaborations. When the smallest set is a singleton, the two notions become equivalent.

2. (Weak) Strict Robustness

We are given a finite set I of players, a finite set A_i of actions of each $i \in I$, and a payoff function $g_i\colon A \to \mathbb{R}$ of each $i \in I$, where we denote $A = \prod_{j \in i} A_i$, $A_{-i} = \prod_{j \neq i} A_j$ etc. as usual. The complete information game $(I, A, \mathbf{g})$ is referred to as $\mathbf{g} = (g_i)_{i \in I}$ whenever no confusion arises.

An elaboration of $\mathbf{g}$ is an incomplete information game consisting of the same sets I and $(A_i)_{i \in I}$ of players and actions as those of $\mathbf{g}$, a countable set T_i of types of each $i \in I$, a common prior $P \in \Delta(T)$, and a bounded payoff function $u_i\colon A \times T \to \mathbb{R}$ of each

$i \in I$, where we denote $\mathbf{u} = (u_i)_{i\in I}$. An elaboration is referred to as $(T, P, \mathbf{u})$.[1] An action distribution $\mu \in \Delta(A)$ is an *equilibrium action distribution* of elaboration $U = (T, P, \mathbf{u})$ if there exists a Bayes-Nash equilibrium $\sigma = (\sigma_i)_{i\in I}$, $\sigma_i \colon T_i \to \Delta(A_i)$, of U that induces μ, i.e., $\mu(a) = \sum_{t\in T} P(t) \prod_{i\in I} \sigma_i(t_i)(a_i)$ for all $a \in A$.

Given an elaboration $(T, P, \mathbf{u})$ of $\mathbf{g}$, for $i \in I$ and $\eta \geq 0$, let $T_i^{g_i,\eta}$ denote the set of all types of player i for which the payoff function differs from g_i by at most η in (conditional) expectation, i.e.,

$$T_i^{g_i,\eta} = \left\{ t_i \in T_i \;\middle|\; \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i) \max_{a\in A} |u_i(a, (t_i, t_{-i})) - g_i(a)| \leq \eta \right\}.$$

Also let $\hat{T}_i^{g_i,\eta}$ denote the set of all types of player i for which the payoff function differs from g_i by at most η with (conditional) probability one, i.e.,

$$\hat{T}_i^{g_i,\eta} = \Big\{ t_i \in T_i \;\Big|\; \max_{a\in A} |u_i(a, (t_i, t_{-i})) - g_i(a)| \leq \eta \\ \text{for all } t_{-i} \in T_{-i} \text{ such that } P(t_{-i}|t_i) > 0 \Big\}.$$

We denote $T^{\mathbf{g},\eta} = \prod_{i\in I} T_i^{g_i,\eta}$ and $\hat{T}^{\mathbf{g},\eta} = \prod_{i\in I} \hat{T}_i^{g_i,\eta}$.

The notion of "nearby" incomplete information games in Morris et al. (2023a) is as follows:

Definition 1 (Morris et al. (2023a))**.** For $\varepsilon \geq 0$ and $\eta \geq 0$, an incomplete information game $(T, P, \mathbf{u})$ is an (ε, η)*-elaboration of* $\mathbf{g}$ if $P(T^{\mathbf{g},\eta}) \geq 1 - \varepsilon$.

The notion employed in Oury and Tercieux (2007), which we call strong (ε, η)-elaboration, is as follows:

Definition 2 (Oury and Tercieux (2007))**.** For $\varepsilon \geq 0$ and $\eta \geq 0$, an incomplete information game $(T, P, \mathbf{u})$ is a *strong* (ε, η)*-elaboration of* $\mathbf{g}$ if $P(\hat{T}^{\mathbf{g},\eta}) \geq 1 - \varepsilon$.

Since $\hat{T}_i^{g_i,\eta} \subset T_i^{g_i,\eta}$ by definition, a strong (ε, η)-elaboration is an (ε, η)-elaboration. Strong $(\varepsilon, 0)$-elaborations and $(\varepsilon, 0)$-elaborations are equivalent, and they are equivalent to ε-elaborations of Kajii and Morris (1997).

Thus, the corresponding robustness notions are as follows:

[1]In Morris et al. (2023a), we employed a formulation that explicitly separates payoff states and belief types, while here we follow Oury and Tercieux (2007), where the payoff functions directly depend on types.

Definition 3 (Kajii and Morris (1997))**.** An action distribution $\mu \in \Delta(A)$ is *KM-robust in* $\mathbf{g}$ if for every $\delta > 0$, there exists $\varepsilon > 0$ such that every $(\varepsilon, 0)$-elaboration of $\mathbf{g}$ has an equilibrium action distribution ν such that $\max_{a\in A} |\nu(a) - \mu(a)| \leq \delta$.

Definition 4 (Morris et al. (2023a))**.** An action distribution $\mu \in \Delta(A)$ is *strictly robust in* $\mathbf{g}$ if for every $\delta > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that every (ε, η)-elaboration of $\mathbf{g}$ has an equilibrium action distribution ν such that $\max_{a\in A} |\nu(a) - \mu(a)| \leq \delta$.

Definition 5 (Oury and Tercieux (2007))**.** An action distribution $\mu \in \Delta(A)$ is *weakly strictly robust in* $\mathbf{g}$ if for every $\delta > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that every strong (ε, η)-elaboration of $\mathbf{g}$ has an equilibrium action distribution ν such that $\max_{a\in A} |\nu(a) - \mu(a)| \leq \delta$.

By definition, weak strict robustness is stronger than KM-robustness and weaker than strict robustness: i.e., if μ is weakly strictly robust, then it is KM-robust, and if it is strictly robust, then it is weakly strictly robust. If μ is weakly strictly robust in $\mathbf{g}$, then it is the action distribution of an essential equilibrium of $\mathbf{g}$ (Wu and Jiang, 1962).

In Morris et al. (2023a), we provided sufficient conditions for strict robustness in terms of unique correlated equilibrium and strict monotone potential (MP) maximizer. Thus, these conditions are also sufficient for weak strict robustness: if $\mathbf{g}$ has a unique correlated equilibrium, then it is the unique weakly strictly robust equilibrium of $\mathbf{g}$; and if $a^* \in A$ is a strict MP maximizer of $\mathbf{g}$ with strict monotone potential v, and either $\mathbf{g}$ or v is supermodular, then the degenerate action distribution on a^* is weakly strictly robust in $\mathbf{g}$.[2] In particular, if a^* is a strictly $\mathbf{p}$-dominant equilibrium in $\mathbf{g}$ for some $\mathbf{p} = (p_i)_{i\in I} \in [0, 1]^I$ with $\sum_{i\in I} p_i < 1$, then it is a strict MP maximizer with a supermodular strict monotone potential, so that it is weakly strictly robust in $\mathbf{g}$.[3]

3. (Strong) Limit Full Implementation by Supermodular Elaborations in Supermodular Games

In what follows, we focus on supermodular games $\mathbf{g}$: for each $i \in I$, A_i is linearly ordered, and for $a_i > a'_i$, $g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})$ is weakly increasing in a_{-i}.

[2] See Morris et al. (2023a, Definition 4) for the definition.

[3] We take this opportunity to correct the typographical errors in the definition of strict $\mathbf{p}$-dominance in Morris et al. (2023a, page 363): the condition should read "$\pi_i(a^*_{-i}) > p_i \Rightarrow \mathrm{br}_i^{g_i}(\pi_i|A_i) = \{a^*_i\}$".

In this section, we describe the results of Oury and Tercieux (2007) on limit full implementation and its implications. An elaboration $(T, P, \mathbf{u})$ is supermodular if $\mathbf{u}(\cdot, t)$ is supermodular for all $t \in T$.

Definition 6. An action distribution $\mu \in \Delta(A)$ is (resp. *strongly*) *limit fully implementable by supermodular elaborations in* $\mathbf{g}$ if for any $\delta > 0$, $\varepsilon > 0$, and $\eta > 0$, there exists a supermodular (resp. strong) (ε, η)-elaboration of $\mathbf{g}$ that has a unique equilibrium action distribution ν, and it satisfies $\max_{a \in A} |\nu(a) - \mu(a)| \leq \delta$.

Let $LFI(\mathbf{g})$ (resp. $\widehat{LFI}(\mathbf{g})$) denote the set of (resp. strongly) limit fully implementable action distributions by supermodular elaborations in $\mathbf{g}$. By definition, $\widehat{LFI}(\mathbf{g}) \subset LFI(\mathbf{g})$, and clearly, these sets are closed and convex.

In our language, Oury and Tercieux (2007) showed the following for all supermodular games $\mathbf{g}$:

- $\widehat{LFI}(\mathbf{g}) \neq \emptyset$. In particular, there exists an action profile $a^* \in A$ such that the degenerate distribution on a^* is strongly limit fully implementable by supermodular elaborations (Oury and Tercieux, 2007, Theorem 1).
- As an implication of this existence result, if an action distribution is weakly strictly robust, then it is degenerate on some pure action profile and is a unique action distribution that is strongly limit fully implementable by supermodular elaborations (Oury and Tercieux, 2007, Corollary 1).
- In particular, any (possibly noise-dependent) global game selection is strongly limit fully implementable by supermodular elaborations. It follows that if an action distribution is weakly strictly robust, then it is a noise-independent global game selection (Oury and Tercieux, 2007, Theorem 2).

Therefore, the statement in Morris et al. (2023a, page 366) that strict robustness implies noise-independent selection in all supermodular games holds true.

4. Equivalence in Binary-Action Supermodular Games

In this section, we restrict our attention to binary-action supermodular (BAS) games. In particular, we will show that strict robustness and weak strict robustness are equivalent in these games.

First, for BAS games, limit full implementability and strong limit full implementability are equivalent and characterized by sequential obedience and reverse sequential obedience.[4] Let $SO^1(\mathbf{g})$ (resp. $SO^0(\mathbf{g})$) denote the set of action distributions that satisfy (resp. reverse) sequential obedience.

Proposition 1. *For any BAS game* $\mathbf{g}$*, we have* $\widehat{LFI}(\mathbf{g}) = LFI(\mathbf{g}) = SO^1(\mathbf{g}) \cap SO^0(\mathbf{g})$.

The inclusion $\widehat{LFI}(\mathbf{g}) \subset LFI(\mathbf{g})$ is by definition; the equality $LFI(\mathbf{g}) = SO^1(\mathbf{g}) \cap SO^0(\mathbf{g})$ follows from the arguments in Morris et al. (2024) and is already reported in Morris et al. (2023a, Proposition A.5). The inclusion $SO^1(\mathbf{g}) \cap SO^0(\mathbf{g}) \subset \widehat{LFI}(\mathbf{g})$ in fact follows from a modification of the construction in Morris et al. (2024); see Appendix A.1.

Combining this result with previous results of Oury and Tercieux (2007) and ours, we now have the following:

Theorem 1. *For any BAS game* $\mathbf{g}$ *and any action distribution* $\mu \in \Delta(A)$*, the following conditions are equivalent:*

(1) μ *is strictly robust in* $\mathbf{g}$.

(2) μ *is weakly strictly robust in* $\mathbf{g}$.

(3) $\{\mu\} = \widehat{LFI}(\mathbf{g})$.

(4) $\{\mu\} = LFI(\mathbf{g})$.

(5) μ *is degenerate on a noise-independent global game selection in* $\mathbf{g}$.

(6) $\{\mu\} = SO^1(\mathbf{g}) \cap SO^0(\mathbf{g})$.

(7) μ *is degenerate on a strict MP maximizer of* $\mathbf{g}$.

The implication (1) $\Rightarrow$ (2) is by definition; (2) $\Rightarrow$ (3) by Oury and Tercieux (2007) as reported in Section 3; (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) by Proposition 1 above; (5) $\Leftrightarrow$ (6) by Morris et al. (2023b);[5] (6) $\Leftrightarrow$ (7) by Oyama and Takahashi (2020) and Morris et al. (2024) (as reported in Morris et al. (2023a, Proposition 3)); (7) $\Rightarrow$ (1) by Morris et al. (2023a, Theorem 1).

Note that the implications (7) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) hold for many-action supermodular games (the last holds since $\widehat{LFI}(\mathbf{g}) \neq \emptyset$ by Oury and Tercieux (2007)). We will show in the next section that (3) $\Rightarrow$ (4) also holds for many-action supermodular games.

Beyond BAS games, the equivalence in Theorem 1 breaks down (note that sequential obedience and reverse sequential obedience are defined only for binary-action games);

[4] See Morris et al. (2023a, Definition 5) for the definition.

[5] Morris et al. (2023a, page 366) misstated that Oyama and Takahashi (2023) showed (5) $\Leftrightarrow$ (7) in BAS games; the correct reference is Morris et al. (2023b).

in particular, (5) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) fail in symmetric two-player three-action supermodular games (Basteck and Daniëls, 2011; Oyama and Takahashi, 2011). It is not known whether strict robustness is strictly stronger than weak strict robustness in general (supermodular) games.

5. (Weak) Strict Robustness to Supermodular Elaborations in Supermodular Games

In this section, for general supermodular games, we show that strict robustness and weak strict robustness are equivalent if robustness is required only with respect to the restricted class of supermodular elaborations. We introduce the relevant definitions as follows.[6]

Definition 7. An action distribution $\mu \in \Delta(A)$ is (resp. *weakly*) *strictly robust to supermodular elaborations in* $\mathbf{g}$ if for every $\delta > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that every supermodular (resp. strong) (ε, η)-elaboration of $\mathbf{g}$ has an equilibrium action distribution ν such that $\max_{a \in A} |\nu(a) - \mu(a)| \leq \delta$.

Definition 8. A set of action distributions $M \subset \Delta(A)$ is (resp. *weakly*) *strictly robust to supermodular elaborations in* $\mathbf{g}$ if it is closed, and for every $\delta > 0$, there exist $\varepsilon > 0$ and $\eta > 0$ such that every supermodular (resp. strong) (ε, η)-elaboration of $\mathbf{g}$ has an equilibrium action distribution ν such that $\max_{a \in A} |\nu(a) - \mu(a)| \leq \delta$ for some $\mu \in M$.

By definition, if an action distribution μ is (weakly) strictly robust to supermodular elaborations, then so is the singleton set $\{\mu\}$, and vice versa. It is immediate that the set of all correlated equilibria of $\mathbf{g}$ is strictly robust (and hence weakly strictly robust) to supermodular (in fact, all) elaborations. Also, it is clear that any (resp. weakly) strictly robust set to supermodular elaborations must include $LFI(\mathbf{g})$ (resp. $\widehat{LFI}(\mathbf{g})$).

The next proposition, first, shows that $LFI(\mathbf{g})$ (resp. $\widehat{LFI}(\mathbf{g})$) is indeed a (resp. weakly) strictly robust set to supermodular elaborations, hence a smallest such set. Second, $LFI(\mathbf{g})$ and $\widehat{LFI}(\mathbf{g})$ share the same largest and smallest elements (with respect to the first-order stochastic dominance order on $\Delta(A)$), which are degenerate distributions.[7]

Proposition 2. *For any supermodular game* $\mathbf{g}$*, the following hold:*

[6] The set-valued version of KM-robustness was studied by Morris and Ui (2005).

[7] For $\mu, \nu \in \Delta(A)$, μ first-order stochastically dominates ν if $\mu(B) \leq \nu(B)$ for all lower sets $B \subset A$ (i.e., sets B such that $a' \in B$, whenever $a \in B$ and $a' \leq a$).

(1) *$LFI(\mathbf{g})$ (resp. $\widehat{LFI}(\mathbf{g})$) is the smallest (resp. weakly) strictly robust set to supermodular elaborations in* $\mathbf{g}$.

(2) *There exist action profiles $\overline{a}, \underline{a} \in A$ such that the degenerate action distribution on $\overline{a}$ (resp. $\underline{a}$) is the largest (resp. smallest) element of both $LFI(\mathbf{g})$ and $\widehat{LFI}(\mathbf{g})$.*

Applied to the case when these sets collapse to a singleton, this proposition immediately implies the following:

Theorem 2. *For any supermodular game* $\mathbf{g}$ *and any action distribution $\mu \in \Delta(A)$, the following conditions are equivalent:*

(1) *μ is strictly robust to supermodular elaborations in* $\mathbf{g}$.

(2) *μ is weakly strictly robust to supermodular elaborations in* $\mathbf{g}$.

(3) $\{\mu\} = \widehat{LFI}(\mathbf{g})$.

(4) $\{\mu\} = LFI(\mathbf{g})$.

In this case, μ is degenerate on some pure action profile.

The proof of Proposition 2, given in Appendix A.2, proceeds as follows. For part (1), suppose that, for sufficiently small $\varepsilon, \eta > 0$, a supermodular (resp. strong) (ε, η)-elaboration of a supermodular game $\mathbf{g}$ is given. Then to the type-agent representation H of a finite-type approximation of U, apply the construction of Oury and Tercieux (2007), to obtain an elaboration V of H that fully implements an action distribution close to some equilibrium action distribution of U. (Here, finite approximation is necessary since the construction of Oury and Tercieux (2007) applies only to supermodular games with finitely many players.) Viewed as an elaboration of $\mathbf{g}$, V is in fact a supermodular (resp. strong) $(2\varepsilon, 2\eta)$-elaboration of $\mathbf{g}$, and hence, it follows that U has an equilibrium action distribution within a neighborhood of $LFI(\mathbf{g})$ (resp. $\widehat{LFI}(\mathbf{g})$).

For part (2) of Proposition 2, take any $\mu, \mu' \in LFI(\mathbf{g})$, and for sufficiently small $\varepsilon, \eta > 0$, let U and U' be supermodular (ε, η)-elaborations of $\mathbf{g}$ that fully implement action distributions ν and ν' close to μ and μ', respectively. Then from U and U', we construct a supermodular strong $(2\varepsilon, C\eta)$-elaboration V (for some constant $C > 0$) such that every equilibrium action distribution of V first-order stochastically dominates both ν and ν'; at least one such distribution lies in a neighborhood of $\widehat{LFI}(\mathbf{g})$ by part (1). By compactness, as $\varepsilon, \eta \to 0$ this implies that $LFI(\mathbf{g})$ has a largest element in $\widehat{LFI}(\mathbf{g})$; since $\widehat{LFI}(\mathbf{g}) \subset LFI(\mathbf{g})$, the two sets share the same largest element. Moreover, if $\mu \in LFI(\mathbf{g})$ is not degenerate on a pure action distribution, applying the above construction with

$\mu = \mu'$ yields some element in $\widehat{LFI}(\mathbf{g})$ that strictly dominates μ, which implies that the (common) largest element must be degenerate. (A symmetric argument applies to the smallest element.)

Finally, for BAS games $\mathbf{g}$, by Proposition 1, the smallest strictly robust set to supermodular elaborations and the weakly strictly robust set to supermodular elaborations coincide, equal to the convex polytope $SO^1(\mathbf{g}) \cap SO^0(\mathbf{g})$. By Theorem 1, the condition that $\mu \in \Delta(A)$ is (weakly) strictly robust to supermodular elaborations in $\mathbf{g}$ is equivalent to the conditions thereof; in particular, (weak) strict robustness to supermodular elaborations is equivalent to that to all elaborations. It remains as an open problem to determine whether these properties extend to many-action supermodular games.

Appendix

A.1. **Proof Proposition 1.** Let $\mathbf{g}$ be a BAS game, where we denote $A_i = \{0, 1\}$. It remains to show that $SO^1(\mathbf{g}) \cap SO^0(\mathbf{g}) \subset \widehat{LFI}(\mathbf{g})$. Let $\mu \in SO^1(\mathbf{g}) \cap SO^0(\mathbf{g})$, i.e., there exist $\rho^+, \rho^- \in \Delta(\Gamma)$ such that

$$\sum_{\gamma \in \Gamma_i} \rho^+(\gamma)(g_i(1, a^1_{-i}(\gamma)) - g_i(0, a^1_{-i}(\gamma))) \geq 0,$$
$$\sum_{\gamma \in \Gamma_i} \rho^-(\gamma)(g_i(0, a^0_{-i}(\gamma)) - g_i(1, a^0_{-i}(\gamma))) \geq 0$$

for all $i \in I$, and $\mu(a) = \rho^+(\{\gamma \in \Gamma \mid a^1(\gamma) = a\}) = \rho^-(\{\gamma \in \Gamma \mid a^0(\gamma) = a\})$ for all $a \in A$.[8] Let $\varepsilon > 0$ and $\eta > 0$ be given. By modifying the construction in Morris et al. (2024, Appendix B.1), we show that there exists a supermodular strong (ε, η)-elaboration of $\mathbf{g}$ that has a unique Bayes-Nash equilibrium, which induces μ.

Let $\zeta > 0$ be sufficiently small that $(1 - \zeta)^{|I|-1} \geq 1 - \varepsilon$, and

$$\sum_{\gamma \in \Gamma_i} (1 - \zeta)^{|I| - n^1(a^1_{-i}(\gamma)) - 1} \rho^+(\gamma)(g_i(1, a^1_{-i}(\gamma)) - g_i(0, a^1_{-i}(\gamma)) + \eta) > 0,$$

for all $i \in I$ such that $\rho^+(\Gamma_i) > 0$ and

$$\sum_{\gamma \in \Gamma_i} (1 - \zeta)^{|I| - n^0(a^0_{-i}(\gamma)) - 1} \rho^-(\gamma)(g_i(0, a^0_{-i}(\gamma)) - g_i(1, a^0_{-i}(\gamma)) + \eta) > 0$$

[8] Γ denotes the set of all sequences of distinct players, and for $i \in I$, Γ_i denotes the set of all sequences in Γ where player i is listed; and for $\gamma \in \Gamma$ and $h = 0, 1$, $a^h(\gamma) \in A$ denotes the action profile such that player i plays action h if and only if i is listed in γ, and for $i \in I$, $a^h_{-i}(\gamma) \in A_{-i}$ denotes the action profile of player i's opponents such that player $j \neq i$ plays action h if and only if j is listed in γ before i.

for all $i \in I$ such that $\rho^-(\Gamma_i) > 0$, where $n^h(a^h_{-i}(\gamma))$, $h = 0, 1$, denotes the number of players playing action h in the action profile $a^h_{-i}(\gamma)$. Then construct the elaboration $(T, P, \mathbf{u})$ of $\mathbf{g}$ as follows:

- the type space of each $i \in I$ is

$$T_i = \{(s_i, a_i) \in \{1, 2, \dots\} \times A_i \mid \mu(\{a_i\} \times A_{-i}) > 0\},$$

- the prior distribution $P \in \Delta(T)$ is given by

$$P(t) = \begin{cases} \zeta(1-\zeta)^m \dfrac{\rho^+(\gamma^+)\rho^-(\gamma^-)}{\mu(a)} & \text{if } \mu(a) > 0 \text{, and there exist } m \in \mathbb{N}, \\ & \gamma^+ \in \Pi(S(a)) \text{, and } \gamma^- \in \Pi(I \setminus S(a)) \\ & \text{such that } s_i = m + \ell(i, \gamma^+) \text{ for all} \\ & i \in S(a) \text{ and } s_i = m + \ell(i, \gamma^-) \text{ for} \\ & \text{all } i \in I \setminus S(a),^{9} \\ 0 & \text{otherwise} \end{cases}$$

  for $t = (s_i, a_i)_{i \in I} \in T$, where $S(a) = \{i \in I \mid a_i = 1\}$, for $S \subset I$, $\Pi(S) \subset \Gamma$ denotes the set of permutations of the players in $S \subset I$, and for $i \in I$ and $\gamma = (i_1, \dots, i_k) \in \Gamma$, $\ell(i, \gamma) = \ell$ if $i = i_\ell$, and

- the payoff function $u_i \colon A \times T \to \mathbb{R}$ of each $i \in I$ is given by

$$u_i(a, t) = \begin{cases} g_i(a) + \eta & \text{if } s_i \geq |I| \text{ and } a_i = a'_i, \\ g_i(a) & \text{if } s_i \geq |I| \text{ and } a_i \neq a'_i, \\ 1 & \text{if } s_i \leq |I| - 1 \text{ and } a_i = a'_i, \\ 0 & \text{if } s_i \leq |I| - 1 \text{ and } a_i \neq a'_i \end{cases}$$

  for $a = (a_j)_{j \in I} \in A$ and $t = (s_j, a'_j)_{j \in I} \in T$.

This is a supermodular strong (ε, η)-elaboration of $\mathbf{g}$: $t_i = (s_i, a_i) \in \hat{T}_i^{g_i, \eta}$ if $s_i \geq |I|$, and thus we have $P(\hat{T}^{\mathbf{g}, \eta}) \geq \sum_{m=|I|-1}^{\infty} \zeta(1-\zeta)^m = (1-\zeta)^{|I|-1} \geq 1 - \varepsilon$.

Then a similar argument as in the proof of Theorem B.1(2) in Morris et al. (2024) shows that action 1 (resp. 0) is uniquely rationalizable for all players of types $t_i = (s_i, a_i)$ with $a_i = 1$ (resp. $a_i = 0$). By construction, the unique rationalizable strategy profile, hence the unique Bayes-Nash equilibrium, induces μ, as desired.

A.2. **Proof of Proposition 2.** In what follows, we let $A_i = \{0, \dots, |A_i| - 1\}$ for each $i \in I$.

The following is our key lemma, which generalizes Oury and Tercieux (2007, Theorem 1) to supermodular elaborations.

[9] For any $t \in T$, there exists at most one such combination of m, γ^+, and γ^-.

Lemma A.1. *Let* $\mathbf{g}$ *be any supermodular game. For any* $\delta > 0$, $\varepsilon > 0$, *and* $\eta > 0$ *and any supermodular (resp. strong)* (ε, η)*-elaboration* U *of* $\mathbf{g}$*, there exist an equilibrium action distribution* μ *of* U *and a supermodular (resp. strong)* $(2\varepsilon, 2\eta)$*-elaboration* U' *of* $\mathbf{g}$ *such that* U' *has a unique equilibrium action distribution* ν*, and it satisfies* $\max_{a \in A} |\nu(a) - \mu(a)| \leq \delta$.

Proof. We prove the statement for strong (ε, η)-elaborations; the proof for (ε, η)-elaborations is analogous.

Let $(I, A, \mathbf{g})$ be a supermodular game. Fix any $\delta > 0$, $\varepsilon > 0$, and $\eta > 0$. Let $U = (I, A, T, P, \mathbf{u})$ be a supermodular strong (ε, η)-elaboration of $(I, A, \mathbf{g})$, where $P(\hat{T}^{\mathbf{g},\eta}) \geq 1 - \varepsilon$. For each $i \in I$, fix an increasing sequence $\{T_i^{(n)}\}_{n=0}^{\infty}$ of finite subsets of T_i such that $\bigcup_{n=0}^{\infty} T_i^{(n)} = T_i$ (where T_i is a countable set by assumption). For each $n \geq 0$, let $U^{(n)} = (I, A, T, P, \mathbf{u}^{(n)})$ be a supermodular elaboration of $\mathbf{g}$ where for each $i \in I$, $u_i^{(n)} \colon A \times T \to \mathbb{R}$ is defined by

$$u_i^{(n)}(a, t) = \begin{cases} u_i(a, t) & \text{if } t_i \in T_i^{(n)}, \\ 1 & \text{if } t_i \notin T_i^{(n)} \text{ and } a_i = 0, \\ 0 & \text{if } t_i \notin T_i^{(n)} \text{ and } a_i \neq 0 \end{cases}$$

for $a = (a_i)_{i \in I}$ and $t = (t_i)_{i \in I}$. This is in effect a game with finitely many types $t_i \in T_i^{(n)}$ for each $i \in I$, while the "dummy types" $t_i \notin T_i^{(n)}$ always play the dominant action $0 \in A_i$. Let $H^{(n)} = (K^{(n)}, B^{(n)}, \mathbf{h}^{(n)})$ be the type-agent representation of $U^{(n)}$, where

- the set of agents is $K^{(n)} = \{(i, t_i) \mid i \in I, t_i \in T_i^{(n)}\}$,
- all agents $k = (i, t_i) \in K^{(n)}$ of player $i \in I$ have a common action set $B_k^{(n)} = A_i$, where $B^{(n)} = \prod_{k \in K^{(n)}} B_k^{(n)}$, and
- the payoff function $h_k^{(n)} \colon B^{(n)} \to \mathbb{R}$ of each $k = (i, t_i) \in K^{(n)}$ is given by

$$h_k^{(n)}(b) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i) u_i((b_{j,t_j})_{j \in I}, (t_i, t_{-i}))$$

  for $b = (b_\ell)_{\ell \in K^{(n)}} \in B^{(n)}$, where we set $b_{j,t_j} = 0$ if $t_j \notin T_j^{(n)}$ in the right-hand side.

By the finiteness of $T_i^{(n)}$'s (hence of $K^{(n)}$) and the supermodularity of $\mathbf{u}(\cdot, t)$ for each $t \in T$, $H^{(n)}$ is a finite supermodular game. Thus we can apply the construction in the proof of Theorem 1 of Oury and Tercieux (2007) to obtain, for each $n \geq 0$, a pure-action Nash equilibrium $b^{(n)} = (b_k^{(n)})_{k \in K^{(n)}} \in B^{(n)}$ of $H^{(n)}$ and a (finite) supermodular elaboration $V^{(n)} = (K^{(n)}, B^{(n)}, (\Omega^{(n)})^{K^{(n)}}, Q^{(n)}, \mathbf{v}^{(n)})$ of $H^{(n)}$ with a unique Bayes-Nash equilibrium where

- the set of players is $K^{(n)}$,

- the set of actions of each $k \in K^{(n)}$ is $B_k^{(n)}$,
- all agents have a common (finite) type space $\Omega^{(n)}$, with a prior distribution $Q^{(n)} \in \Delta((\Omega^{(n)})^{K^{(n)}})$,
- with some $\hat{\Omega}^{(n)} \subset \Omega^{(n)}$ such that $Q^{(n)}((\hat{\Omega}^{(n)})^{K^{(n)}}) \geq 1 - \frac{\varepsilon}{2}$ and some $c^{(n)} \in \mathbb{R}$ such that $|c^{(n)}| \max_{i \in I}(|A_i| - 1) \leq \eta$, the payoff function $v_k^{(n)} \colon B^{(n)} \times (\Omega^{(n)})^{K^{(n)}} \to \mathbb{R}$ of each $k = (i, t_i) \in K^{(n)}$ is given by

$$v_k^{(n)}(b, \omega) = \begin{cases} h_k^{(n)}(b) + c^{(n)} b_k & \text{if } \omega_k \in \hat{\Omega}^{(n)}, \\ 1 & \text{if } \omega_k \notin \hat{\Omega}^{(n)} \text{ and } b_k = b_k^{(n)}, \\ 0 & \text{if } \omega_k \notin \hat{\Omega}^{(n)} \text{ and } b_k \neq b_k^{(n)} \end{cases}$$

  for $b = (b_\ell)_{\ell \in K^{(n)}} \in B^{(n)}$ and $\omega = (\omega_\ell)_{\ell \in K^{(n)}} \in (\Omega^{(n)})^{K^{(n)}}$, and
- in the unique Bayes-Nash equilibrium, each $k \in K^{(n)}$ plays the pure action $b_k^{(n)}$ for all $\omega_k \in \Omega^{(n)}$.

For each $n \geq 0$ and each $i \in I$, define the pure strategy $\sigma_i^{(n)} \colon T_i \to A_i$ by $\sigma_i^{(n)}(t_i) = b_{i,t_i}^{(n)}$ if $t_i \in T_i^{(n)}$ and $\sigma_i^{(n)}(t_i) = 0$ if $t_i \notin T_i^{(n)}$. By construction, $\sigma^{(n)} = (\sigma_i^{(n)})_{i \in I}$ is a Bayes-Nash equilibrium of $U^{(n)}$. Since the set of functions from T_i to A_i is compact in the product topology, $\{\sigma^{(n)}\}$ has a subsequence that converges pointwise to some σ^*, which is a Bayes-Nash equilibrium of U. Let $N \geq 0$ be such that $P(T^{(N)}) \geq 1 - \frac{\varepsilon}{2}$, where $T^{(N)} = \prod_{i \in I} T_i^{(N)}$, and $\max_{a \in A} |\sigma_P^{(N)}(a) - \sigma_P^*(a)| \leq \delta$ (where $\sigma_P^{(N)}$ and σ_P^* are the action distributions induced by $\sigma^{(N)}$ and σ^*, respectively).

We then translate the incomplete information game $V^{(N)} = (K^{(N)}, B^{(N)}, (\Omega^{(N)})^{K^{(N)}}, Q^{(N)}, \mathbf{v}^{(N)})$ to an elaboration $U' = (I, A, T', P', \mathbf{u}')$ of $(I, A, \mathbf{g})$ as follows:

- the type space of each $i \in I$ is $T_i' = T_i \times \Omega^{(N)}$, where $T' = \prod_{i \in I} T_i'$,
- the prior distribution $P' \in \Delta(T')$ is given by

$$P'(t') = P((t_i)_{i \in I}) Q^{(N)}(\{(o_k)_{k \in K^{(N)}} \in (\Omega^{(N)})^{K^{(N)}} \mid o_{i,t_i} = \omega_i \text{ for all } i \in I\})$$

  for $t' = (t_i, \omega_i)_{i \in I} \in T'$, and
- the payoff function $u_i' \colon A \times T' \to \mathbb{R}$ of each $i \in I$ is given by

$$u_i'(a, t') = \begin{cases} u_i(a, (t_j)_{j \in I}) + c^{(N)} a_i & \text{if } t_i \in T_i^{(N)} \text{ and } \omega_i \in \hat{\Omega}^{(N)}, \\ 1 & \text{if } (t_i \notin T_i^{(N)} \text{ or } \omega_i \notin \hat{\Omega}^{(N)}) \text{ and } a_i = \sigma_i^{(N)}(t_i), \\ 0 & \text{if } (t_i \notin T_i^{(N)} \text{ or } \omega_i \notin \hat{\Omega}^{(N)}) \text{ and } a_i \neq \sigma_i^{(N)}(t_i) \end{cases}$$

  for $a = (a_j)_{j \in I} \in A$ and $t' = (t_j, \omega_j)_{j \in I} \in T'$.

By construction, U' is supermodular, and it has the same set of Bayes-Nash equilibria as $V^{(N)}$, and hence the pure-strategy profile $\sigma' = (\sigma'_i)_{i\in I}$ defined by $\sigma'_i(t_i,\omega_i) = \sigma_i^{(N)}(t_i)$ for all $t_i \in T_i$ and all $\omega_i \in \Omega^{(N)}$ is a unique Bayes-Nash equilibrium of U' and induces $\sigma_P^{(N)}$. Also we have

$$P'\left(\prod_{i\in I}((\hat{T}_i^{g_i,\eta}\cap T_i^{(N)})\times\hat{\Omega}^{(N)})\right) \geq P\left(\prod_{i\in I}(\hat{T}_i^{g_i,\eta}\cap T_i^{(N)})\right) Q^{(N)}((\hat{\Omega}^{(N)})^{K^{(N)}}) \geq 1-2\varepsilon,$$

and for any $(t_i,\omega_i) \in (\hat{T}_i^{g_i,\eta}\cap T_i^{(N)})\times\hat{\Omega}^{(N)}$, we have

$$\begin{aligned}&\max_{a\in A}|u'_i(a,(t_j,\omega_j)_{j\in I}) - g_i(a)|\\ &\leq \max_{a\in A}|u_i(a,(t_j,\omega_j)_{j\in I}) - g_i(a)| + |c^{(N)}|(|A_i|-1) \leq 2\eta\end{aligned}$$

for all $(t_j,\omega_j)_{j\neq i} \in T'_{-i}$ such that $P'((t_j,\omega_j)_{j\neq i}|t_i,\omega_i) > 0$ (hence $P(t_{-i}|t_i) > 0$). Hence, U' is a strong $(2\varepsilon, 2\eta)$-elaboration of $(I, A, \mathbf{g})$. This completes the proof of Lemma A.1 (with $\mu = \sigma_P^*$ and $\nu = \sigma'_{P'}$). □

Proof of Proposition 2. For an elaboration U, we write $E(U) \subset \Delta(A)$ for the set of equilibrium action distributions of an elaboration U. For $M \subset \Delta(A)$ and $\delta > 0$, write $B_\delta(M) = \{\nu \in \Delta(A) \mid \max_{a\in A}|\nu(a) - \mu(a)| \leq \delta \text{ for some } \mu \in M\}$.

(1) We prove the statement for strict robustness; the proof for weak strict robustness is analogous. Let $\mathbf{g}$ be a supermodular game. For $\varepsilon > 0$ and $\eta > 0$, write $E^*(\varepsilon,\eta) = \{\mu \in \Delta(A) \mid \{\mu\} = E(U) \text{ for some } (\varepsilon,\eta)\text{-elaboration of } \mathbf{g}\}$. By definition, $\mathit{LFI}(\mathbf{g}) = \bigcap_{\varepsilon,\eta>0}\overline{E^*(\varepsilon,\eta)}$.

Take any $\delta > 0$. By compactness, we can take $\varepsilon > 0$ and $\eta > 0$ such that $E^*(2\varepsilon, 2\eta) \subset B_{\frac{\delta}{2}}(\mathit{LFI}(\mathbf{g}))$. Let U be any (ε,η)-elaboration of $\mathbf{g}$. By Lemma A.1, we have $B_{\frac{\delta}{2}}(E(U)) \cap E^*(2\varepsilon,2\eta) \neq \emptyset$. Thus, $E(U) \cap B_\delta(\mathit{LFI}(\mathbf{g})) \neq \emptyset$, as desired.

(2) By compactness and the inclusion $\widehat{\mathit{LFI}}(\mathbf{g}) \subset \mathit{LFI}(\mathbf{g})$, the claim for the largest element follows from Lemma A.2 below; the argument for the smallest element is symmetric. □

Below, $\geq$ denotes the first-order stochastic dominance order on $\Delta(A)$ or $\Delta(A_i)$, $i \in I$.

Lemma A.2. *Let $\mathbf{g}$ be any supermodular game.*

(i) *For any $\mu, \mu' \in \mathit{LFI}(\mathbf{g})$, there exists $\mu'' \in \widehat{\mathit{LFI}}(\mathbf{g})$ such that $\mu'' \geq \mu$ and $\mu'' \geq \mu'$.*

(ii) *For any $\mu \in \mathit{LFI}(\mathbf{g})$ that is not degenerate on some pure action profile, there exists $\mu'' \in \widehat{\mathit{LFI}}(\mathbf{g})$ such that $\mu'' \neq \mu$ and $\mu'' \geq \mu$.*

Proof. (i) Let $\mathbf{g}$ be a supermodular game, and let $\mu, \mu' \in \mathit{LFI}(\mathbf{g})$. Take any $\delta > 0$. By part (1) of Proposition 2, we can take $\varepsilon > 0$ and $\eta > 0$ such that $E(V) \cap B_\delta(\widehat{\mathit{LFI}}(\mathbf{g})) \neq \emptyset$ for any supermodular strong $(2\varepsilon, 2\max_{i\in I}(|A_i|-1)\eta)$-elaboration V of $\mathbf{g}$. Let $U = (T, P, \mathbf{u})$ and $U' = (T'P', \mathbf{u}')$ be supermodular (ε, η)-elaborations of $\mathbf{g}$ such that $E(U) = \{\nu\}$, $E(U') = \{\nu'\}$, $\nu \in B_\delta(\mu)$, and $\nu' \in B_\delta(\mu')$ for some ν and ν'. Define the elaboration $U'' = (T'', P'', \mathbf{u}'')$ of $\mathbf{g}$ as follows:

- the type space of each $i \in I$ is $T''_i = T_i \times T'_i$, where $T'' = \prod_{i\in I} T''_i$,
- the prior distribution $P'' \in \Delta(T'')$ is given by

$$P''(t'') = P((t_i)_{i\in I})P'((t'_i)_{i\in I})$$

  for $t'' = (t_i, t'_i)_{i\in I} \in T''$, and
- the payoff function $u''_i \colon A \times T'' \to \mathbb{R}$ of each $i \in I$ is given by

$$u''_i(a, t'') = \begin{cases} g_i(a) + 2\eta a_i & \text{if } t''_i \in T_i^{g_i,\eta} \times T_i'^{g_i,\eta}, \\ 1 & \text{if } t''_i \notin T_i^{g_i,\eta} \times T_i'^{g_i,\eta} \text{ and } a_i = |A_i| - 1, \\ 0 & \text{if } t''_i \notin T_i^{g_i,\eta} \times T_i'^{g_i,\eta} \text{ and } a_i \neq |A_i| - 1 \end{cases}$$

  for $a = (a_j)_{j\in I}$ and $t'' = (t''_j)_{j\in I}$.

By construction, U'' is a supermodular strong $(2\varepsilon, 2\max_{i\in I}(|A_i|-1)\eta)$-elaboration of $\mathbf{g}$: $P''(\hat{T}''^{\mathbf{g}, 2\max_{i\in I}(|A_i|-1)\eta}) \geq P''(\prod_{i\in I}(T_i^{g_i,\eta} \times T_i'^{g_i,\eta})) = P(T^{\mathbf{g},\eta})P'(T'^{\mathbf{g},\eta}) \geq 1 - 2\varepsilon$. Thus, by the choice of ε and η, U'' has an equilibrium action distribution ν'' in $B_\delta(\widehat{\mathit{LFI}}(\mathbf{g}))$.

Let Σ_i, Σ'_i, and Σ''_i denote the sets of player i's strategies in U, U', and U'', respectively, with the usual notation $\Sigma = \prod_{i\in I}\Sigma_i$, $\Sigma_{-i} = \prod_{j\neq i}\Sigma_j$, etc. Let $\mathrm{BR}_i \colon \Sigma_{-i} \times T_i \to A_i$, $\mathrm{BR}'_i \colon \Sigma'_{-i} \times T'_i \to A_i$, and $\mathrm{BR}''_i \colon \Sigma''_{-i} \times T''_i \to A_i$ denote the pure best response correspondences of player i in U, U', and U'', respectively.

Claim 1. *For any $i \in I$ and any $\sigma_{-i} \in \Sigma_{-i}$, $\sigma'_{-i} \in \Sigma'_{-i}$, and $\sigma''_{-i} \in \Sigma''_{-i}$, if $\sigma''_{-i}((t_j, t'_j)_{j\neq i}) \geq \sigma_{-i}(t_{-i})$ and $\sigma''_{-i}((t_j, t'_j)_{j\neq i}) \geq \sigma'_{-i}(t'_{-i})$ for all $t_{-i} = (t_j)_{j\neq i} \in T_{-i}$ and $t'_{-i} = (t'_j)_{j\neq i} \in T'_{-i}$, then $\min \mathrm{BR}''_i(\sigma''_{-i})(t_i, t'_i) \geq \min \mathrm{BR}_i(\sigma_{-i})(t_i)$ and $\min \mathrm{BR}''_i(\sigma''_{-i})(t_i, t'_i) \geq \min \mathrm{BR}'_i(\sigma'_{-i})(t'_i)$ for all $t_i \in T_i$ and $t'_i \in T'_i$.*

Proof. We only prove the former inequality. Suppose that $\sigma''_{-i}((t_j, t'_j)_{j\neq i}) \geq \sigma_{-i}(t_{-i})$ for all $t_{-i} \in T_{-i}$ and $t'_{-i} \in T'_{-i}$. It suffices to consider types $t_i \in T_i^{g_i,\eta}$ and $t'_i \in T_i'^{g_i,\eta}$. Denote $a_i = \min \mathrm{BR}_i(\sigma_{-i})(t_i)$, and consider any $b_i < a_i$. Then we have

$$\sum_{t_{-i}\in T_{-i}, t'_{-i}\in T'_{-i}} P''((t_j, t'_j)_{j\neq i}|t_i, t'_i)[u''_i(a_i, \sigma''_{-i}((t_j, t'_j)_{j\neq i})) - u''_i(b_i, \sigma''_{-i}((t_j, t'_j)_{j\neq i}))]$$

$$= \sum_{t_{-i}\in T_{-i}, t'_{-i}\in T'_{-i}} P(t_{-i}|t_i)P'(t'_{-i}|t'_i)$$
$$\times [g_i(a_i, \sigma''_{-i}((t_j, t'_j)_{j\neq i})) - g_i(b_i, \sigma''_{-i}((t_j, t'_j)_{j\neq i})) + 2\eta(a_i - b_i)]$$

$$\geq \sum_{t_{-i}\in T_{-i}, t'_{-i}\in T'_{-i}} P(t_{-i}|t_i)P'(t'_{-i}|t'_i)$$
$$\times [g_i(a_i, \sigma_{-i}(t_{-i})) - g_i(b_i, \sigma_{-i}(t_{-i})) + 2\eta(a_i - b_i)]$$

$$\geq \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)[g_i(a_i, \sigma_{-i}(t_{-i})) - g_i(b_i, \sigma_{-i}(t_{-i})) + 2\eta]$$
$$\geq \sum_{t_{-i}\in T_{-i}} P(t_{-i}|t_i)[u_i(a_i, \sigma_{-i}(t_{-i})) - u_i(b_i, \sigma_{-i}(t_{-i}))] > 0,$$

where the first inequality holds by supermodularity, and the third inequality holds since $t_i \in T_i^{g_i,\eta}$. Thus, $\min \mathrm{BR}''_i(\sigma''_{-i})(t_i, t'_i) \geq a_i$ as claimed. □

By Claim 1, every Bayes-Nash equilibrium of U'', in particular the Bayes-Nash equilibrium that induces ν'', is weakly larger than both the unique Bayes-Nash equilibria of U and U' which induce ν and ν', respectively, which implies that $\nu''(B) \leq \nu(B)\nu'(B)$ for all lower sets $B \subset A$.

Finally, let $\delta \to 0$. By compactness, we have some $\mu'' \in \widehat{LFI}(\mathbf{g})$ such that $\mu''(B) \leq \mu(B)\mu'(B)$ for all lower sets $B \subset A$, which implies that $\mu'' \geq \mu$ and $\mu'' \geq \mu'$.

(ii) Let $\mu \in LFI(\mathbf{g})$, and suppose that μ is not degenerate, so that $0 < \mu(B) < 1$ for some lower set $B \subset A$. Then apply the above argument with $\mu = \mu'$ to obtain some $\mu'' \in \widehat{LFI}(\mathbf{g})$ such that $\mu'' \geq \mu$, where we have $\mu''(B) \leq \mu(B)^2 < \mu(B)$, and hence $\mu'' \neq \mu$. □

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