Finite Markov Chains I

Daisuke Oyama

Topics in Economic Theory

November 12, 2014

Classification of States

Let $\{X_t\}_{t=0}^{\infty}$ be a finite-state discrete-time Markov chain represented by an $n \times n$ stochastic matrix P, with the state space denoted by $S = \{0, 1, \ldots, n-1\}$.

Definition 1

- State i *has access* to state j, denoted $i \to j$, if $(P^k)_{ij} > 0$ for some $k = 0, 1, 2, \ldots$, where $P^0 = I$.
- States i and j *communicate*, denoted $i \leftrightarrow j$, if $i \to j$ and $j \to i$.

Lemma 1

The binary relation $\leftrightarrow$ is an equivalent relation: it is

1. *reflexive, i.e., $i \leftrightarrow i$ for all $i \in S$;*
2. *symmetric, i.e., if $i \leftrightarrow j$, then $j \leftrightarrow i$; and*
3. *transitive, i.e., if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.*

Definition 2

$C \subset S$ is a *communication class* of $\{X_t\}$, or of P, if it is an equivalent class of $\leftrightarrow$.

Definition 3

$\{X_t\}$, or P, is *irreducible* if it has only one communication class.

It is *reducible* if it is not irreducible.

Definition 4

A state i is *recurrent* if $i \to j$ implies $j \to i$.

It is *transient* if it if not recurrent.

Lemma 2

For any communication class C and any states $i, j \in C$,
i is recurrent if and only if j is recurrent.

- Thus, recurrence is a property of a communication class.

Definition 5

A communication class C is a *recurrent class* if
it contains a recurrent state.

It is *transient* if it if not recurrent.

- A recurrent class is also called a *closed communication class*.

Definition 6 (Stokey and Lucas, 11.1)

$E \subset S$ is an *ergodic set* if

- $\sum_{j \in E} P_{ij} = 1$, and
- If $F \subset E$ and $\sum_{j \in F} P_{ij} = 1$, then $F = E$.

Lemma 3

C is a recurrent class if and only if it is an ergodic set.

Equivalent Definitions by Graph-Theoretic Concepts

- A *directed graph* $\Gamma = (V, E)$ consists of
  - a nonempty set V of *nodes* (or *vertices*), and
  - a set $E \subset V \times V$ of *edges* (or *directed edges* or *arcs*).
- A subgraph of a directed graph Γ is a directed graph (V', E') such that $V' \subset V$ and $E' \subset E$.
- A *path* is a sequence of nodes $(v_0, v_1, \dots, v_k)$ such that $(v_i, v_{i+1}) \in E$.

  (This is often called a *walk*, and a path often refers to a *simple walk*, a walk where the nodes are all distinct, except possibly for v_0 and v_k.)

  For each $v \in V$, (v) is also considered to be a path.

  We define the *length* of a path $(v_0, v_1, \dots, v_k)$ to be k.

- State i *has access* to state j, denoted $i \to j$, if there is a path that starts with i and terminates with j.
- States i and j *communicate*, denoted $i \leftrightarrow j$, if $i \to j$ and $j \to i$.
- $\leftrightarrow$ is an equivalent relation, and thus partitions V into equivalent classes.
- A *strongly connected component* (SCC) of Γ is a subgraph (C, E_C) of Γ such that C is an equivalent class of $\leftrightarrow$ and $E_C = E \cap (C \times C)$.
- Γ is *strongly connected* if V constitutes a single equivalent class of $\leftrightarrow$.

- The *condensation* of Γ is the directed graph $(V/_{\leftrightarrow}, \tilde{E})$ where
  - $V/_{\leftrightarrow}$ is the quotient set (the set of equivalent classes) of $\leftrightarrow$, and
  - $(C, C') \in \tilde{E}$ if and only if $C \neq C'$ and there exist $i \in C$ and $j \in C'$ such that $i \to j$.
- The condensation is *acyclic*, i.e., it has no path $(V_0, \ldots, V_k)$ such that $V_0 = V_k$.

  Therefore, it has at least one *sink node*, a node $C \in V/_{\leftrightarrow}$ such that $(C, C') \notin \tilde{E}$ for all $C' \in V/_{\leftrightarrow}$.

- Given a stochastic matrix P with state space S, let $\Gamma(P) = (S, E)$ be the directed graph such that $(i, j) \in E$ if and only if $P_{ij} > 0$.

Observation 1

For all $k = 0, 1, \ldots$ and all $i, j \in S$, the following are equivalent:

- $P^k_{ij} > 0$;
- *there is a path of $\Gamma(P)$ of length k from i to j.*

Thus, the two equivalent relations are equal.

Observation 2

- *$C \subset S$ is a communication class of P if and only if C is the set of nodes of a strongly connected component of $\Gamma(P)$.*
- *P is irreducible if and only if $\Gamma(P)$ is strongly connected.*
- *$C \subset S$ is a recurrent class if and only if C constitutes a sink node of the condensation of $\Gamma(P)$.*

Observation 3

Any Markov chain, or stochastic matrix, has at least one recurrent class.

Stationary Distributions

Definition 7

$x \in \mathbb{R}^n_+$ is a *stationary distribution* of $\{X_t\}$, or of P, if $x'P = x'$ and $x'\mathbf{1} = 1$, where $\mathbf{1} \in \mathbb{R}^n$ is the vector of ones.

Proposition 4

Any Markov chain, or stochastic matrix, has at least one stationary distribution.

Proposition 5

Let P be an irreducible stochastic matrix.

- *P has a unique stationary distribution.*
- *The unique stationary distribution is strictly positive.*

Proposition 6

For any stationary distribution x, $x_i = 0$ for any transient state i.

Corollary 7

- *For any stationary distribution x and any recurrent class C, if $\operatorname{supp}(x) \cap C \neq \emptyset$, then $C \subset \operatorname{supp}(x)$, and $x|_C / \|x|_C\|$ is the unique stationary distribution of $P|_C$.*
- *Any stationary distribution is a convex combination of these stationary distributions.*

Corollary 8

P has a unique stationary distribution if and only if it has a unique recurrent class.

Periodicity

Definition 8

$d \in \mathbb{Z}_{++}$ is the *period* of state i if it is the greatest common divisor of all k's such that $(P^k)_{ii} > 0$.

Lemma 9

For any communication class C and any states $i, j \in C$, i has period d if and only if has period d.

- Thus, recurrence is a property of a communication class.

Definition 9

- The period of a recurrent class is the period of any state in that class.
- The period of $\{X_t\}$, or of P, is the least common multiple of the periods of the recurrent classes.
- $\{X_t\}$, or of P, is *aperiodic* if its period is one.

Proposition 10

- *For any stochastic matrix P, $\lim_{t\to\infty}(1/t)\sum_{s=0}^{t-1} P^s$ exists, and each row of it is a stationary distribution.*
- *If P is aperiodic, then $\lim_{t\to\infty} P^t$ exists, and each row of it is a stationary distribution.*
- *If P in addition has only one recurrent class, then*

$$\lim_{t\to\infty} P^t = \begin{pmatrix} x' \\ \vdots \\ x' \end{pmatrix},$$

where x is the unique stationary distribution of P.

- Suppose that an irreducible Markov chain has period d.
- Fix any state, say state 0.
- For each $m = 0, \ldots, , d - 1$, let S_m be the set of states i such that $(P^{kd+m})_{0i} > 0$ for some k.
- These sets $S_0, \ldots, , S_{d-1}$ constitute a partition of S and are called the *cyclic classes*.
- For each S_m and each $i \in S_m$, we have $\sum_{j \in S_{m+1}} P_{ij} = 1$, where $S_d = S_0$.