Analysis in Metric Space I

Daisuke Oyama

Topics in Economic Theory

October 1, 2014

Metric Spaces

Definition 1

For a nonempty set S, a function $\rho\colon S \times S \to \mathbb{R}$ is a *metric* or *distance* on S if for any $x, y, z \in S$,

0. $\rho(x, y) \geq 0$,
1. $\rho(x, y) = 0$ if and only if $x = y$,
2. $\rho(x, y) = \rho(y, x)$, and
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

The pair (S, ρ) is called a *metric space*.

Often we say that S is a metric space with a metric or distance ρ. If ρ is understood, we simply say that S is a metric space.

- Condition 0 actually follows from conditions 1–3.

Normed Vector Spaces

Definition 2

For a nonempty vector space V (over $\mathbb{R}$), a function $\|\cdot\|: V \to \mathbb{R}$ is a *norm* on V if for any $x, y \in V$ and $\gamma \in \mathbb{R}$,

0. $\|x\| \geq 0$,
1. $\|x\| = 0$ if and only if $x = 0$,
2. $\|\gamma x\| = |\gamma|\|x\|$, and
3. $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(V, \|\cdot\|)$ is called a *normed vector space* or *normed space*.

Often we say that V is a normed vector space with a norm $\|\cdot\|$.
If $\|\cdot\|$ is understood, we simply say that V is a normed vector space.

- Condition 0 actually follows from conditions 1–3.

- Normed spaces are a special case of metric spaces.

Proposition 1 (Exercise 3.1.3)

Let $(V, \|\cdot\|)$ *be a normed vector space, and define the function* $\rho\colon V \times V \to \mathbb{R}$ *by* $\rho(x, y) = \|x - y\|$. *Then* (V, ρ) *is a metric space.*

Examples (p-Norm)

- For $\mathbb{R}^k$ and $p \geq 1$, let $\|\cdot\|_p \colon \mathbb{R}^k \to \mathbb{R}$ be defined by

$$\|x\|_p = \left(\sum_{i=1}^{k} |x_i|^p \right)^{1/p},$$

  and $\|\cdot\|_\infty \colon \mathbb{R}^k \to \mathbb{R}$ be defined by

$$\|x\|_\infty = \max_{i=1,\dots,k} |x_i|.$$

- For any nonempty set U,
  let bU be the set of all bounded functions $f \colon U \to \mathbb{R}$
  (functions such that $\sup_{x \in U} |f(x)| < \infty$).

  This is a vector space with $(f+g)(x) = f(x) + g(x)$ and $(\gamma f)(x) = \gamma f(x)$.

  Let

$$\|f\|_\infty = \sup_{x \in U} |f(x)|.$$

- Let ℓ_p be the set of all functions $x\colon \mathbb{N} \to \mathbb{R}$ such that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty.$$

This is a vector space with $x + y = (x_1 + y_1, x_2 + y_2, \ldots)$ and $\gamma x = (\gamma x_1, \gamma x_2, \ldots)$.

Let

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

- For $\ell = \infty$, let $\ell_\infty = b\mathbb{N}$, and $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

Sequences

- For a nonempty set S, a *sequence* in S is a function from $\mathbb{N}$ to S.

  A sequence is denoted by (x_n).

  For $A \subset S$, we write $(x_n) \subset A$ if $x_n \in A$ for all $n \in \mathbb{N}$.

Definition 3

Let (S, ρ) be a metric space.

A sequence $(x_n) \subset S$ is said to *converge* to $\bar{x} \in S$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho(x_n, \bar{x}) < \varepsilon \text{ for all } n \geq N.$$

In this case, we write $\lim_{n\to\infty} x_n = \bar{x}$, or $x_n \to \bar{x}$;
and we say that (x_n) is *convergent*.

$\bar{x}$ is called the *limit* of (x_n).

Proposition 2 (Theorem 3.1.6)

A sequence in a metric space has at most one limit.

- Let (S, ρ) be a metric space. Let

$$B(r; x) = \{y \in S \mid \rho(y, x) < r\}.$$

- A subset $E \subset S$ is *bounded* if $E \subset B(n; x)$ for some $x \in S$ and $n \in \mathbb{N}$.
- A sequence $(x_n) \subset S$ is bounded if $\{x_n \in S \mid n \in \mathbb{N}\}$ is bounded.

Proposition 3 (Exercise 3.1.9)

Any convergent sequence in a metric space is bounded.

Subsequences

Definition 4

For a sequence $(x_n) \subset S$, a sequence $(y_n) \subset S$ is a *subsequence* of (x_n) if there exists a strictly increasing function $f \colon \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{f(n)}$ for all $n \in \mathbb{N}$.

We write (x_{n_k}) to denote a subsequence of (x_n).

Proposition 4

Let (S, ρ) be a metric space.

For $(x_n) \subset S$ and $x \in S$,

1. *$x_n \to x$ if and only if every subsequence of (x_n) converges to x;*
2. *$x_n \to x$ if and only if every subsequence of (x_n) has a subsequence that converges to x.*

Continuous Functions

Definition 5

Let S and T be metric spaces, and A a nonempty subset of S.

A function $f\colon A \to T$ is *continuous* at $a \in A$ if
for any $(x_n) \subset A$ with $x_n \to a$, we have $f(x_n) \to f(a)$.

f is continuous on $B \subset A$ if f is continuous at all $a \in B$.

f is continuous if it is continuous on A.

Proposition 5 (Example 3.1.12)

Let (S, ρ) is a metric space, and let $\bar{x}$ be any point in S.

Then the function $\rho(\cdot, \bar{x})\colon S \ni x \mapsto \rho(x, \bar{x}) \in \mathbb{R}$ is continuous.