Contagion and Uninvadability in Local Interaction Games: The Bilingual Game and General Supermodular Games

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Topics in Economic Theory

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Paper

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Introduction

- A game is played on a large network.

  Players interact with their neighbors.

- Changes in actions of a small set of players can have large impacts through contagion.

- When does contagion occur?
  Which action?
  In what kind of network?

Introduction

In this paper, we conduct the following exercise:

- Fix a two-player game.

  Analyze long-run behavior of best response dynamics played on various networks.

- Key phenomena are:
  - contagion
  - uninvadability: no other action is contagious in *any* network

- Understand how the payoff structure affects contagion/uninvadability.

Introduction

... And also the following "reverse" exercise:

- Understand how the network structure (network topology) affects contagion.
- More specifically:
  Fix a network and a parameterized class of games, and find the parameter range in which contagion occurs.
- Networks are classified by these parameter ranges.

  E.g., a network is more contagion-inducing than another.

Introduction

We consider the bilingual game,
a 3×3 game obtained by adding a *bilingual option*
to a 2×2 coordination game.

- May be of interest itself.
- Refine the analysis by Morris (2000) based on 2×2 games:

  Weakly (indeed strictly, in some cases) more detailed analysis of network topologies.
- Simple enough to obtain a full characterization of contagion/uninvadability.

Literature

This game has been studied by

[1] Goyal and Janssen (1997)

Consider a circle (circular network).

[2] Immorlica, Kleinberg, Mahdian, and Wexler (2007)

Consider "regular" graphs for the case where Pareto-dominance and risk-dominance coincide.

[3] Easley and Kleinberg (2010)
Networks, Crowds, and Markets: Reasoning about a Highly Connected World

Plan of the Talk

- Definition of contagion and uninvadability
- Review: 2×2 Coordination Games
- Bilingual Game—A 3×3 Game
- "Reverse Exercise": Comparison of Networks
- (Implications in Incomplete Information Games)

A Network

- $\mathcal{X}$: a countably infinite set of nodes
- $P\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$: interaction weights
  - $P(x, x) = 0$
  - $P(x, y) = P(y, x)$
  - $\sum_{x,y} P(x, y) = \infty$
  - $0 < \sum_y P(x, y) < \infty$

For example,

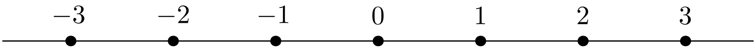


Figure: Simple linear network

- $\mathcal{X} = \mathbb{Z}$, $P(x, y) = \begin{cases} 1 & \text{if } |x - y| = 1, \\ 0 & \text{otherwise} . \end{cases}$

A Game on a Network

- $u\colon A \times A \to \mathbb{R}$: a symmetric two-player game
- $\sigma\colon \mathcal{X} \to A$: an action configuration
- Player x chooses a to maximize

$$\sum_y u(a, \sigma(y))P(x, y) \propto \sum_y u(a, \sigma(y))P(y|x)$$

  - $P(y|x) = \frac{P(x,y)}{\sum_{y'} P(x,y')}$: normalized weight

Sequential Best Response Dynamics

$(\sigma^t)_{t=0}^{\infty}$: a best response sequence

- For each t, there exists at most one x, denoted by x^t, such that $\sigma^t(x^t) \neq \sigma^{t-1}(x^t)$.
- $\sigma^t(x^t)$ is a best response against σ^{t-1}.
- If $\lim_{t\to\infty} \sigma^t(x) = s$,
  then for all $T \geq 0$, $s \in BR(\sigma^t|x)$ for some $t \geq T$.

Most of our results go through with simultaneous best responses.

Contagion/Uninvadability in Network Games

- a^* is contagious in network $(\mathcal{X}, P)$ if:
  there exists a finite set of players s.t.
  if this set of players initially plays action a^*,
  then the whole population will eventually play a^*
  in any best response sequence.
  - a^* is contagious if it is contagious in *some* network.
- a^* is uninvadable if:
  for all networks,
  if a^* is played by almost all players,
  then it continues to be played by almost all players
  in any best response sequence.

Review: 2×2 Coordination Games

$$u = \begin{array}{c|c|c|} & A & B \\ \hline A & 11 & 0 \\ \hline B & 3 & 10 \\ \hline \end{array}$$

- Both A and B: Nash equilibria
- A: Pareto-dominant
- B: risk-dominant
  - B is the best response against the opponent's mixed action $\frac{1}{2}A + \frac{1}{2}B$.

Contagion and Uninvadability in 2×2 Games

- The risk-dominant action B is contagious.
  (Demonstrated by many papers.)
- B is also uninvadable. (Morris 2000)
  - The risk-dominated action A cannot be contagious.

The Proof of Contagion

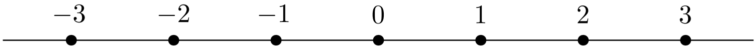


Figure: Simple linear network

- B spreads contagiously from two consecutive nodes.

The Proof of Uninvadability

The proof is based on potential/Lyapunov functions.

- Replace payoff function u by its potential v

$u =$

	A	B
A	11	0
B	3	10

by

$v =$

	A	B
A	11	3
B	3	13

- v is symmetric, and satisfies

$$v(a,b) - v(a',b) = u(a,b) - u(a',b)$$

  In particular, u and v are best response equivalent:
  incentives to choose A or B is the same between u and v.

- v is maximized at (B,B).

- Define the Lyapunov function by

$$V(\sigma) = \frac{1}{2} \sum_{x,y \in \mathcal{X}} P(x,y) \big[v(\sigma(x), \sigma(y)) - 13 \big]$$

- In general, $-\infty \leq V(\sigma) \leq 0$
- Suppose that "almost all" players play B in period 0.
  More formally, the initial weight on non-(B, B) pairs is finite:

$$\frac{1}{2} \sum_{(\sigma^0(x), \sigma^0(y)) \neq (B,B)} P(x,y) < \infty$$

- $V(\sigma^0) > -\infty$.

- Recall that only $x^t \in \mathcal{X}$ changes action in period t.

$$
\begin{aligned}
& V(\sigma^t) - V(\sigma^{t-1}) \\
&= \frac{1}{2} \sum_{y \in \mathcal{X}} P(x^t, y)(v(\sigma^t(x^t), \sigma^t(y)) - v(\sigma^{t-1}(x^t), \sigma^{t-1}(y))) \\
&\quad + \frac{1}{2} \sum_{y \in \mathcal{X}} P(y, x^t)(v(\sigma^t(y), \sigma^t(x^t)) - v(\sigma^{t-1}(y), \sigma^{t-1}(x^t))) \\
&= \sum_{y \in \mathcal{X}} P(x^t, y)(v(\sigma^t(x^t), \underbrace{\sigma^t(y)}_{=\sigma^{t-1}(y)}) - v(\sigma^{t-1}(x^t), \sigma^{t-1}(y))) \\
&\geq 0.
\end{aligned}
$$

- $V(\sigma^t) \geq V(\sigma^{t-1}) \geq \cdots \geq V(\sigma^0) > -\infty$.
- The weight on non-(B, B) pairs is bounded from above.
- Thus B is uninvadable.

Reverse Exercise: Contagion Thresholds

We quantify the "power" of a network

- $u_p =$

	A	B
A	1	0
B	$1-p$	$1-p$

- Contagion threshold of $(\mathcal{X}, P)$:

$$\xi(\mathcal{X}, P) := \sup\{p : B \text{ is contagious in } (\mathcal{X}, P) \text{ for } u_p\}$$

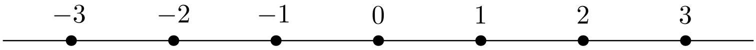


Figure: Simple linear network

- The contagion threshold of the simple linear network is $1/2$.

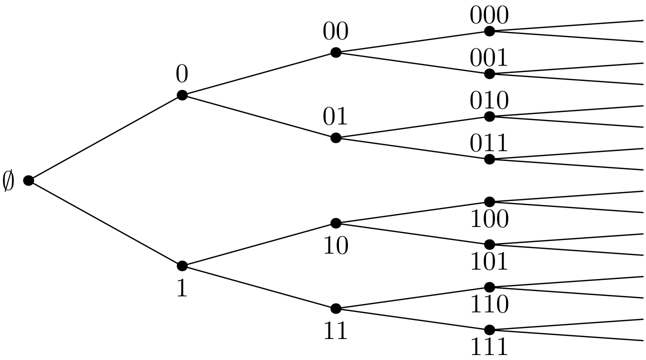


Figure: "Tree"

- The contagion threshold of the tree is $1/3$.

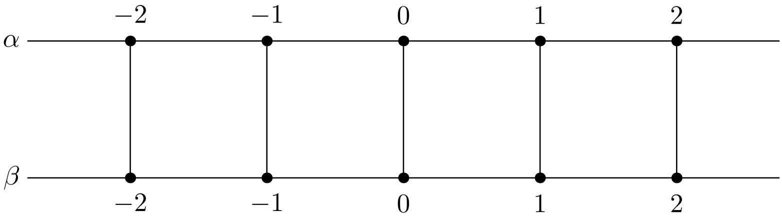


Figure: "Ladder"

- The contagion threshold of the ladder is $1/3$.

- All networks are “linearly ordered” according to the contagion threshold.
- The contagion threshold is at most $1/2$.
- The contagion threshold is maximized at the simple linear network.

(Morris 2000)

Summary for 2×2 Games

- The risk-dominant action is contagious in the simple linear network.
- The risk-dominant action is uninvadable.
- All networks are "linearly ordered" according to the contagion threshold.
  - The simple linear network is most "powerful".

We will see that these results no longer hold for 3×3 games.

Bilingual Game

AB: “bilingual option” or “compatible technology” with cost $e > 0$

	A	AB	B
A	a	a	b
AB	$a - e$	$a - e$	$d - e$
B	c	d	d

$a > c$, $d > b$

We assume:

- $a > d$ $\cdots$ A Pareto-dominates B.
- $a - c < d - b$ $\cdots$ B pairwise risk-dominates A.
- $c \leq d$ $\cdots$ The game is supermodular.

Focus on the case where $c - b < a - c$.

Contagion in the Linear Network

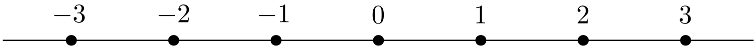


Figure: Linear network

- If e is small enough ($e < (a-d)/2$) so that $br(\frac{1}{2}[A] + 0[AB] + \frac{1}{2}[B]) = AB$:

	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$t=0$	$\cdots$	B	A	A	A	B	B	B	$\cdots$
$t=1$	$\cdots$	B	A	A	A	AB	B	B	$\cdots$
$t=2$	$\cdots$	B	A	A	A	AB	AB	B	$\cdots$
$t=3$	$\cdots$	B	A	A	A	A	AB	B	$\cdots$

- $\Rightarrow$ A is contagious in this network.
- But, this network is not “critical”.

- What is the largest value of e for which A is contagious?
- What is the smallest value of e for which B is contagious?

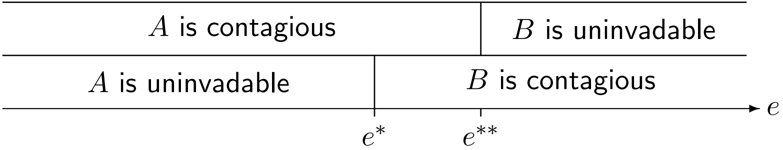
(By definition, "contagious" = "contagious in *some* network".)

Our Result 1

	A	AB	B
A	a	a	b
AB	$a-e$	$a-e$	$d-e$
B	c	d	d

(A: Pareto-dominant; B: risk-dominant)

Result 1. In the class of all networks,



$$e^* = \frac{(a-d)(d-b)}{2(c-b)}, \; e^{**} = \frac{(a-d)(d-b)(a-c)}{(c-b)(d-b)+(a-c)(a-d)}.$$

Contagion of B when $e > e^*$

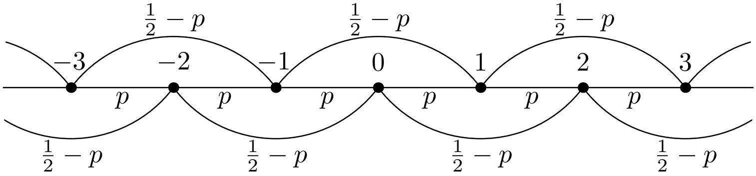


Figure: Contagion of action B

There exists $p \in (0, 1/2)$ s.t.

- $br\big(\frac{1}{2}A + pAB + (\frac{1}{2} - p)B\big) = AB$ or B
- $br\big((\frac{1}{2} - p)A + pAB + \frac{1}{2}B\big) = B$

Contagion of A when $e < e^{**}$

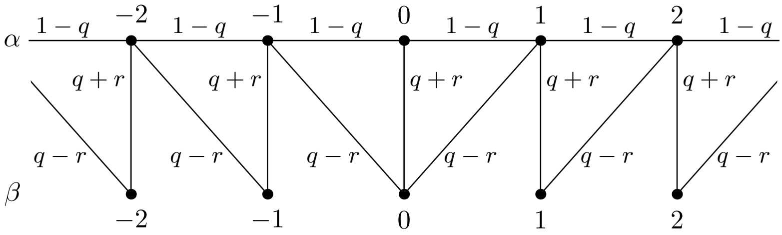
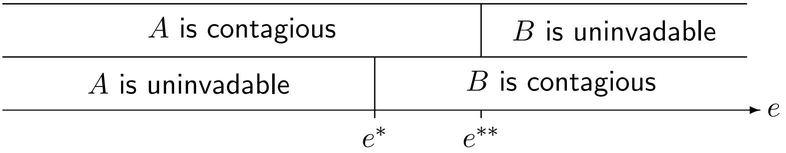


Figure: Contagion of action A

Then there exist $q, r \in (0,1)$, $q \geq r$, s.t.

- $br\left(\frac{1+q}{2}A + \frac{1-q}{2}B\right) = A$
- $br\left(\frac{q+r}{2q}AB + \frac{q-r}{2q}B\right) = A$
- $br\left(\frac{1-r}{2}A + \frac{1+r}{2}B\right) = AB$



A is contagious
B is uninvadable
A is uninvadable
B is contagious
e
e^*
e^{**}

Uninvadability

- The uninvadability result is proved by finding
  a monotone potential function of the bilingual game.

- Construction of such a potential function is ad hoc.

Restricted Domains

What if we restrict attentions to a class of "simple" networks?

- "Linear/lattice network":

  Players are located on a line (or a lattice) and have translation-invariant interactions with neighbors.

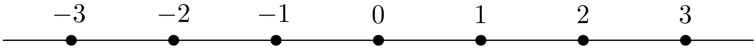


Figure: Linear network

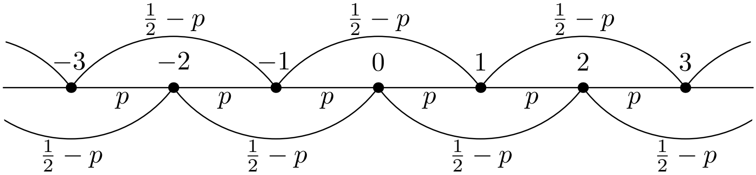


Figure: Linear network

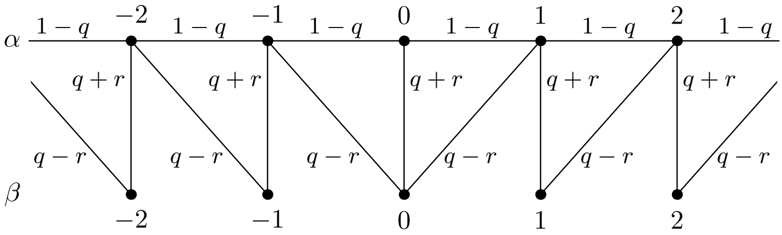
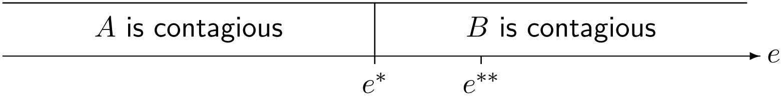


Figure: Non-linear network

Our Result 2

Result 2. *In the class of linear/lattice networks*,



If $e^* < e < e^{**}$, then A is contagious only in non-linear networks. Linear/lattice networks are not enough to determine contagion.

"Reverse Exercise"

- In the class of 2-action coordination games, Tree and Ladder have the same contagion threshold $1/3$.
- The bilingual game can differentiate these two networks.

Tree vs. Ladder

There is a range of payoff parameter values such that

- $br(\frac{2}{3}A + \frac{1}{3}AB) = A$
- $br(\frac{2}{3}A + \frac{1}{3}B) = AB$
- $br(\frac{1}{3}A + \frac{1}{3}AB + \frac{1}{3}B) = B$

$\Rightarrow$ B is not contagious in Tree, but is contagious in Ladder.

Ladder

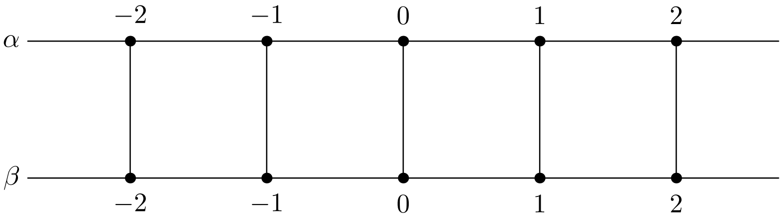


Figure: "Ladder"

- In the above parameter range,
  B is contagious in Ladder.

Tree

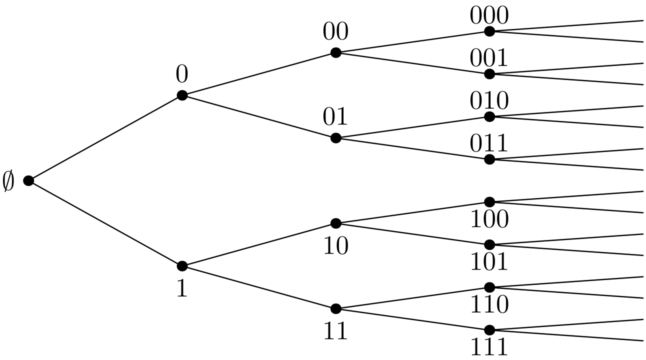


Figure: "Tree"

- In the above parameter range,
  B is not contagious in Tree.

Bundling

- An action that is contagious in Ladder may not be contagious in Tree.
- But the opposite is always true.
If an action is contagious in Tree, then it is contagious in Ladder.
- More generally, if one can "bundle" nodes in a network $\mathcal{X}$ and obtain another network $\mathcal{X}'$,
then contagion is (at least weakly) more likely to occur in $\mathcal{X}'$ than in $\mathcal{X}$

Our Result 3

- $\varphi\colon \mathcal{X} \to \mathcal{X}'$ is a weight-preserving node identification if φ is onto (i.e., $\varphi(\mathcal{X}) = \mathcal{X}'$) and it preserves weights:

$$P'(\varphi(x), y') = \sum_{y \in \varphi^{-1}(y')} P(x, y) \qquad \forall x \in \mathcal{X} \setminus E, y' \in \mathcal{X}'.$$

  with a finite set E of exceptional nodes.

- Result 3. *If there is a weight-preserving node identification from $(\mathcal{X}, P)$ to $(\mathcal{X}', P')$, then $(\mathcal{X}', P')$ is more contagion-inducing than $(\mathcal{X}, P)$, i.e., for any action a^* in any symmetric supermodular game, if a^* is contagious in $(\mathcal{X}, P)$, then a^* is contagious in $(\mathcal{X}', P')$.*

Examples

- Ladder is more contagion-inducing than Tree.

  (φ does *not* preserve interaction weights at the "root"; so we need "exceptional nodes" in the definition of φ.)

  As shown, Ladder is *strictly* more contagion-inducing than Tree.

- Linear networks are weakly (indeed strictly) more contagion-inducing than multidimensional lattices.

- (The existence of weight-preserving node identifications is sufficient, but not necessary for two networks to be comparable.)

Implications in incomplete information games

- Incomplete information games and local interaction games are formally equivalent and belong to the general class of "interaction games"
  (Morris 1997, 2000).
  - Incomplete information games:
    each type interacts with a subset of types and payoffs are given by the weighted sum of those from the interactions.
  - Local interaction games:
    each node interacts with a set of neighbors and payoffs are given by the (weighted) sum of those from the interactions.
- Results in one class of games readily translate into the other.

See:

- Oyama and Takahashi,
  On the Relationship between Robustness to Incomplete Information and Noise-Independent Selection in Global Games
  (*Journal of Mathematical Economics* 47, 2011)

Conclusion

We studied the Bilingual Game:

- Simple enough to obtain a full characterization of contagion/uninvadability.
- Allows us to understand rich structures of networks by analyzing contagion phenomena;

  in particular, there are networks (e.g., Ladder/Tree) that are differentiated by the analysis with the bilingual game but not by that with 2×2 games.

Open Issues

- Given a "realistic" network, identify what happens (whether contagion of A or B occurs, etc.).
- Random networks.
- More general classes of games.
- Heterogeneity among players.
- $\ldots$