Global Games

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Topics in Economic Theory

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Papers

- Carlsson, H. and E. van Damme (1993). "Global Games and Equilibrium Selection," Econometrica 61, 989-1018.
- Frankel, D., S. Morris, and A. Pauzner (2003). "Equilibrium Selection in Global Games with Strategic Complementarities," Journal of Economic Theory 108, 1-44.

Setting (FMP)

Global game $G(\kappa)$

- Players: $I = \{1, \dots, |I|\}$
- Actions of player i: $A_i = \{0, 1, \dots, n_i\}$
- State: $\theta \in \mathbb{R} \sim$ continuous density ϕ, connected support
- Payoffs of player i: $u_i(a, \theta)$
  - $\Delta u_i(a_i, a_i', a_{-i}, \theta) = u_i(a_i, a_{-i}, \theta) - u_i(a_i', a_{-i}, \theta)$
- Signal of player i: $x_i = \theta + \kappa\varepsilon_i$
  - $(\varepsilon_1, \dots, \varepsilon_{|I|}) \sim$ continuous joint density f, support contained in $[-\frac{1}{2}, \frac{1}{2}]^I$
  - Independent of θ

Assumptions

1. Strategic complementarities:

   For all $i \in I$,

   if $a_i \geq a_i'$ and $a_{-i} \geq a_{-i}'$, then
   $\Delta u_i(a_i, a_i', a_{-i}, \theta) \geq \Delta u_i(a_i, a_i', a_{-i}', \theta)$ for all θ.

2. Dominance regions:

   There exist $\underline{\theta}$ and $\overline{\theta}$ such that for all $i \in I$,

   - if $\theta \leq \underline{\theta}$, then $\Delta u_i(0, a_i, a_{-i}, \theta) > 0$ for all $a_i \neq 0$ and all a_{-i}; and
   - if $\theta \geq \overline{\theta}$, then $\Delta u_i(n_i, a_i, a_{-i}, \theta) > 0$ for all $a_i \neq n_i$ and all a_{-i}.

3. Strict state monotonicity:

   There exists $K_0 > 0$ such that for all $i \in I$ and all a_{-i},

   if $a_i \geq a_i'$ and $\theta \geq \theta'$, then

$$\Delta u_i(a_i, a_i', a_{-i}, \theta) - \Delta u_i(a_i, a_i', a_{-i}, \theta') \geq K_0(a_i - a_i')(\theta - \theta').$$

4. Payoff continuity:

   For all $i \in I$ and all a, $u_i(a, \theta)$ is continuous in θ.

Two Players, Two Actions (Carlsson and van Damme)

- $I = \{1, 2\}$, $A_1 = A_2 = \{0, 1\}$
- Payoffs:

	0	1
0	0	0
1	$-p_i(\theta)$	$1 - p_i(\theta)$

- $p_i(\theta)$: continuous, strictly decreasing
- $p_i(\theta) > 1$ if $\theta \leq \underline{\theta}$ ($\Rightarrow$ action 0 is dominant)
  $p_i(\theta) < 0$ if $\theta \geq \overline{\theta}$ ($\Rightarrow$ action 1 is dominant)
- For θ such that $0 < p_i(\theta) < 1$, $i = 1, 2$,
  $(0, 0)$ and $(1, 1)$ are strict Nash equilibria.

- Risk-dominance in asymmetric 2×2 coordination games:
  - $(1,1)$ risk-dominant at θ
    $\iff p_1(\theta)p_2(\theta) < (1-p_1(\theta))(1-p_2(\theta))$
    $\iff p_1(\theta) + p_2(\theta) < 1$
  - $(0,0)$ risk-dominant at θ
    $\iff p_1(\theta)p_2(\theta) > (1-p_1(\theta))(1-p_2(\theta))$
    $\iff p_1(\theta) + p_2(\theta) > 1$
- Let θ^* be the unique θ such that

$$p_1(\theta) + p_2(\theta) = 1$$

  - $(1,1)$ risk-dominant at θ iff $\theta > \theta^*$
  - $(0,0)$ risk-dominant at θ iff $\theta < \theta^*$

Simplifying Assumptions

- Uniform prior:

  ϕ uniform on some $[a, b]$ (sufficiently large)

- Private values:

  Payoffs $u_i(a, x_i)$, depending on signal x_i (rather than θ)

Equilibrium Selection

Theorem 1 (Two-player, two-action case)

In the limit as $\kappa \to 0$*, an essentially unique strategy profile survives iterative dominance; and*
it plays the risk-dominant equilibrium.

Posterior Beliefs

- f^κ: joint density of $(\kappa\varepsilon_1, \kappa\varepsilon)$
  $(f^\kappa(z_1, z_2) = \frac{1}{\kappa^2} f(\frac{z_1}{\kappa}, \frac{z_2}{\kappa}))$
- Conditional density (for x_1, x_2 away from the boundary):

$$
\begin{aligned}
f_1^\kappa(x_2|x_1) &= \frac{\int f^\kappa(x_1-\theta, x_2-\theta)\phi(\theta)d\theta}{\iint f^\kappa(x_1-\theta, x_2-\theta)dx_2\phi(\theta)d\theta} \\
&= \frac{\int f^\kappa(x_1-\theta, x_2-\theta)d\theta}{\iint f^\kappa(x_1-\theta, x_2-\theta)dx_2 d\theta} \\
&= \int f^\kappa(x_1-\theta, x_2-\theta)d\theta \\
f_2^\kappa(x_1|x_2) &= \cdots = \int f^\kappa(x_1-\theta, x_2-\theta)d\theta
\end{aligned}
$$

- Density of $\kappa\varepsilon_1 - \kappa\varepsilon_2$:

$$\psi^\kappa(y) = \int f^\kappa(z + y, z) dz$$

- Thus,

$$\begin{aligned} f_1^\kappa(x_2 | x_1) = f_2^\kappa(x_1 | x_2) &= \int f^\kappa(x_1 - \theta, x_2 - \theta) d\theta \\ &= \int f^\kappa(z + x_1 - x_2, z) dz \\ &= \psi^\kappa(x_1 - x_2) \end{aligned}$$

- Therefore,

$$
\begin{aligned}
P(x_2 \geq \xi_2 | x_1 = \xi_1) &= \int_{x_2 \geq \xi_2} \psi^\kappa(\xi_1 - x_2) dx_2 \\
&= \int_{x_1 \leq \xi_1} \psi^\kappa(x_1 - \xi_2) dx_1 \\
&= 1 - P(x_1 \geq \xi_1 | x_2 = \xi_2)
\end{aligned}
$$

or

$$
P(x_2 \geq \xi_2 | x_1 = \xi_1) + P(x_1 \geq \xi_1 | x_2 = \xi_2) = 1 \qquad (*)
$$

Iterative Dominance

- By Dominance regions, player i observing a signal above some threshold $\overline{\xi}_i^1$ play 1.
- Assuming that player j with signals above $\overline{\xi}_j^1$ play 1,

  by Action monotonicity and State monotonicity, player i observing a signal above some threshold $\overline{\xi}_i^2$ play 1, where $\overline{\xi}_i^2 \leq \overline{\xi}_i^1$.
- $\dots$
- We have $\overline{\xi}_i^1 \geq \overline{\xi}_i^2 \geq \cdots \searrow \overline{\xi}_i$.
- Similarly, from below we have $\underline{\xi}_i^1 \leq \underline{\xi}_i^2 \leq \cdots \nearrow \underline{\xi}_i$.
- All these depend on κ;

  write the limits as $\overline{\xi}_i(\kappa)$ and $\underline{\xi}_i(\kappa)$.

- By continuity, player 1 with signal $\overline{\xi}_1(\kappa)$ when opponents play 1 above $\overline{\xi}_2(\kappa)$ and 0 below $\overline{\xi}_2(\kappa)$ must be indifferent between playing 1 and 0:

$$P(x_2 \geq \overline{\xi}_2(\kappa) | x_1 = \overline{\xi}_1(\kappa)) = p_1(\overline{\xi}_1(\kappa))$$

- Similarly,

$$P(x_1 \geq \overline{\xi}_1(\kappa) | x_2 = \overline{\xi}_2(\kappa)) = p_2(\overline{\xi}_2(\kappa))$$

- In particular, it must be that $|\overline{\xi}_1(\kappa) - \overline{\xi}_2(\kappa)| < \kappa$.
- By $(*)$, we have

$$p_1(\overline{\xi}_1(\kappa)) + p_2(\overline{\xi}_2(\kappa)) = 1.$$

- Let $(\overline{\xi}^*, \overline{\xi}^*)$ be any limit point of $(\overline{\xi}_1(\kappa), \overline{\xi}_2(\kappa))$ as $\kappa \to 0$.
- By continuity, we have

$$p_1(\overline{\xi}^*) + p_2(\overline{\xi}^*) = 1.$$

- Therefore, we must have $\overline{\xi}^* = \theta^*$.
- It follows that $\overline{\xi}_1(\kappa)$ and $\overline{\xi}_2(\kappa)$ both converge to θ^* as $\kappa \to 0$.
- Symmetrically, $\underline{\xi}_1(\kappa)$ and $\underline{\xi}_2(\kappa)$ both converge to θ^* as $\kappa \to 0$.
- Hence, in the limit as $\kappa \to 0$, the game is dominance solvable, and the surviving strategy profile plays $(1, 1)$ if $\theta > \theta^*$ and $(0, 0)$ if $\theta < \theta^*$.

Many Players, Many Actions (FMP)

- Limit Uniqueness
- Noise dependence of the surviving strategy profile
- Sufficient conditions for noise independence
  - Two-player two-action case
  - ...

Limit Uniqueness

Theorem 2

$G(\kappa)$ has an essentially unique strategy profile surviving iterative dominance in the limit as $\kappa \to 0$.
More precisely, there exists an increasing pure strategy profile s^* such that if for each $\kappa > 0$, s^κ is a pure strategy profile surviving iterative dominance in $G(\kappa)$, then $\lim_\kappa s^\kappa(x) = s^*(x)$ for all x but except possibly for finitely many discontinuous points of s^*.

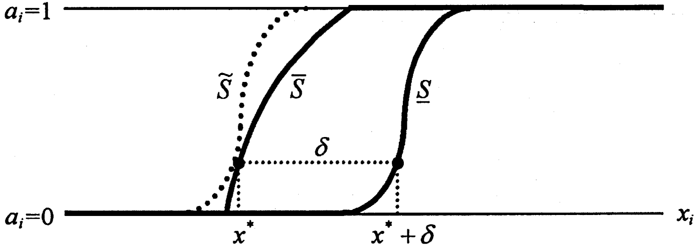

Fig. 4.

(From FMP 2003, p.8)

Global Game Selections

- Let θ^* be a point at which s^* is continuous.
- $s^*(\theta^*)$ is a Nash equilibrium of the complete information game $(u_i(\cdot, \theta^*))_{i \in I}$.
- Say that $s^*(\theta^*)$ is a *global game selection* in $(u_i(\cdot, \theta^*))_{i \in I}$.

Noise Dependence

- In general, global game selection is noise dependent.

  There are games in which different equilibria are selected under different noise distributions f_i.

- On the other hand, global game selection does not depend on the prior distribution ϕ, and how $(u_i(\cdot, \theta))_{i \in I}$ behaves for $\theta \neq \theta^*$ (as long as the assumptions are satisfied).

Noise Independence

- In certain games, global game selection is independent of the noise distribution.
- Carlsson and van Damme (1993) showed that in 2×2 coordination games, a risk-dominant equilibrium is a noise-independent global game selection.
- FMP show:

  if the game has a "local potential" (LP) function and satisfies own-action concavity, then the LP-maximizer is a noise-independent global game selection.
- More generally,

  if the game has a "monotone potential" (MP) function, then the MP-maximizer is a noise-independent global game selection.

Table 1
Noise (in)dependence in supermodular games.

Symmetric games			
actions:	2 each	3 each	4 each
2 players	✓[a]	✓[c]	×[b]
3 players	✓[b]	×[d]	
n players	✓[b]		

Asymmetric games			
actions:	2 each	2 by n	3 each
2 players	✓[a]	✔[g]	×[c]
3 players	×[e]	n/a	
n players	×[f]	n/a	

✓ Always noise independent. × Counterexample to noise independence exists. For empty cells noise dependence follows from an example in smaller games.

[a] Carlsson and Van Damme [6].
[b] Frankel, Morris and Pauzner [10].
[c] Basteck and Daniëls [1].
[d] Basteck et al. [2].
[e] Carlsson [4].
[f] Corsetti et al. [7].
[g] This paper, see Section 5: Two-player games with 2 by n actions.

(From Basteck, Daniels, and Heinemann 2013, p.2629)